ABOUT MIN-PLUS PRODUCT FORMS

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ABSTRACT. We study here the min-plus analogues of Jackson networks of queues and show that the corresponding geodesic problems on $\mathbb{Z}^m$ can be reduced to minimal cost flow problems on complete graphs having m nodes. In some particular cases, these flow problems can be solved explicitly giving formulae analogue, in the min-plus algebra, to the standard product forms.

Nous étudions les analogues min-plus des réseaux de files d'attente Jacksonien et montrons que les problèmes correspondants de géodésique sur $\mathbb{Z}^m$ se ramènent à des problèmes de flot à coût minimal sur des graphes complets à m noeuds. Dans certain cas, ces problèmes de flot peuvent être résolus explicitement, donnant, dans l'algèbre min-plus, les analogues des formes-produit standards.

1. INTRODUCTION

We can associate to a network of m queues a random walk on $\mathbb{Z}^m$. The min-plus analogue of a random walk is a decision walk where to each transition — which corresponds to a decision — is associated a cost instead of a probability. In this min-plus context, the dynamic programming equation plays the role of the Kolmogorov equation. It is well known that the invariant measure of a Jackson network can be computed explicitly see [17, 5, 26, 10]. The min-plus analogue consists in computing the optimal cost to go from a node x to a node y in the state space. It is a kind of geodesic problem on $\mathbb{Z}^m$ with a field of admissible displacements corresponding to the admissible routings of the network. We show that this geodesic problem can be solved by a standard flow problem under the hypothesis of shift invariance of the transition costs. Moreover, for some particular ends (x, y) of the geodesic, an explicit formula, analogue to the standard product form, giving the minimal distance between x and y, is obtained.

2. SOME MIN-PLUS ALGEBRA AND NOTATIONS

A *semiring* $\mathcal{K}$ is a set endowed with two operations denoted $\oplus$ and $\otimes$ where $\oplus$ is associative, commutative with zero element denoted ε, $\otimes$ is associative, admits a unit element denoted e, and distributes over $\oplus$; zero is absorbing ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$ for all $a \in \mathcal{K}$). This semiring is *commutative* when $\otimes$ is commutative. A module on a semiring is called a *semimodule*. A *dioid* $\mathcal{K}$ is a semiring which is idempotent ($a \oplus a = a, \ \forall a \in \mathcal{K}$). A [commutative, resp. idempotent] *semifield* is a [commutative, resp. idempotent] semiring whose nonzero elements are invertible.

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The set $\mathbb{R} \cup \{+\infty\}$ endowed with the two operations $\oplus = \min$, $\otimes = +$, is denoted $\mathbb{R}_{\min}$. This structure is traditionally called min-plus algebra. It is an idempotent semifield with $\varepsilon = +\infty$ and $e = 0$. The structure $\mathbb{R}_{\min}$, completed with $-\infty$, with the convention $+\infty - \infty = +\infty$, is a dioid denoted $\overline{\mathbb{R}}_{\min}$.

We denote $\mathcal{M}_p$ the dioid of (p, p)-matrices with entries in the semiring $\mathcal{K}$. The matrix product in $\mathcal{M}_p$ is

$$[AB]_{ij} \stackrel{\text{def}}{=} [A \otimes B]_{ij} \stackrel{\text{def}}{=} \min_k [A_{ik} + B_{kj}] \,.$$

All the entries of the zero matrix of $\mathcal{M}_p$ are $+\infty$. The diagonal entries of the identity matrix of $\mathcal{M}_p$ are 0, the other entries being $+\infty$.

With a matrix M in $\mathcal{M}_n(\mathcal{K})$, we associate a *precedence graph* $\mathcal{G}(M) = (\mathcal{N}, \mathcal{P})$ with nodes $\mathcal{N} = \{1, 2, \cdots, n\}$, and arcs $\mathcal{P} = \{xy \mid x, y \in \mathcal{N},\ M_{xy} \neq \varepsilon\}$. The number M_{xy}, when it is nonzero, is called the weight of the arc xy.

A path π, of length l, with origin x and end y, is an ordered set of nodes $\pi = \pi_0\pi_1 \cdots \pi_l$ with $\pi_0 = x$ and $\pi_l = y$, and $\pi_i\pi_{i+1} \in \mathcal{P}$ for all $i = 0, \cdots, l-1$. The couple $\pi_i\pi_{i+1}$ are called the arcs of π and the π_i its nodes. The *length* of the path π is denoted $|\pi|$. The couple xy of the *ends* of π is denoted $\langle\pi\rangle$. When the two ends of π are equal one says that π is a circuit. The *weight* of π, denoted $\pi(M)$, is the $\otimes$-product of the weights of its arcs. For example we have $xyz(M) = M_{xy} \otimes M_{yz}$.

The set of all paths with ends xy and length l is denoted $\mathcal{P}^l_{xy}$. The paths of length 0 are the nodes $\mathcal{P}^0 = \mathcal{N}$. Then, $\mathcal{P}^*_{xy}$ is the set of all paths with ends xy and $\mathcal{P}^*$ the set of all paths. We have :

$$\mathcal{P}^* \stackrel{\text{def}}{=} \bigcup_{l=0}^{\infty} \mathcal{P}^l \,.$$

For $\rho \subset \mathcal{P}$, $\langle\rho\rangle$ is the set of the ends of the paths of ρ. Then denoting $\mathcal{P}_N$ the set of arcs of the graph associated to the matrix N we have the following trivial accessibility results :

PROPOSITION 1. *For $M \in \mathcal{M}_n$ we have :*

$$\mathcal{P}_{M^k} = \langle\mathcal{P}^k\rangle, \quad \mathcal{P}_{M^*} = \langle\mathcal{P}^*\rangle.$$

For $\rho \subset \mathcal{P}^*$ one define :

$$\rho(M) \stackrel{\text{def}}{=} \bigoplus_{\pi \in \rho} \pi(M) \,,$$

which is the infimum of the weights of all the paths belonging to ρ.

We denote

$$M^* \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} M^i \,,$$

which exists if we accept entries in $\overline{\mathbb{R}}_{\min}$. Then, we have the following interpretation of the matrix product in $\mathcal{M}_n$.

PROPOSITION 2. *For $M \in \mathcal{M}_n$ we have*

$$\mathcal{P}^l_{xy}(M) = (M^l)_{xy}, \quad \mathcal{P}^*_{xy}(M) = (M^*)_{xy} \,. \tag{1}$$

The matrix M^* has no entries equal to $-\infty$ iff there is no circuits with negative weight in $\mathcal{P}$.

More details about min-plus algebra can be found in [6, 22, 15, 18, 16].

3. DECISION CALCULUS

A min-plus probability calculus has been developed in [25, 13, 7, 1, 2, 3, 4, 12]. Let us recall the most elementary facts. On a set U, a *cost* $c : U \mapsto \mathbb{R}_{\min}$ satisfying $\bigoplus_{u\in U} c(u) = e$ is given. It is called a min-plus *probability density*. A subset A of U, seen as a decision set, is the analogue of an event. The cost of the decision set A is $c(A) \stackrel{\text{def}}{=} \bigoplus_{u\in A} c(u)$, it corresponds to the probability of an event. Then, the functions $X : U \mapsto \mathbb{R}$ are called decision variables by analogy with random variables. They induce the costs $c_X(x) \stackrel{\text{def}}{=} \bigoplus_{X(u)=x} c(u)$ on $\mathbb{R}$. Following this analogy all the standard notions of the probability calculus can be introduced.

The min-plus Markov chain is called a Bellman chain, it is defined by a transition cost matrix $M \in \mathcal{M}_n$ satisfying $Me = e$ where e denotes the column of e of size n. Then, given an initial cost, which is a line vector c^0 satisfying $c^0 e = e$, we can define a cost c, on the set of paths $\mathcal{P}$, by $c(\pi) = c^0_{\pi_0}\pi(M)$ for all $\pi \in \mathcal{P}^l$ and $l \in \mathbb{N}$. Then, the analogue of the forward Kolmogorov equation is the forward Bellman equation $c^n = c^{n-1} \otimes M$, c^0 given. It gives the marginal cost, for the Bellman chain $X^n(\pi) \stackrel{\text{def}}{=} \pi_n$, to be in state (node) $x \in \mathcal{N}$ at time n.

If a transition cost matrix satisfies

$$M_{xy} = M_{yx} > 0,\ M_{xx} \geq 0,\ \forall y \neq x \in \mathcal{N}\ ,$$

then the matrix M^*_{xy} defines a metric. Indeed, we have $M^*_{xx} = 0$ and $M^*_{xy} \leq M^*_{xz} + M^*_{zy}$ by definition of the matrix product and the fact that $M^*M^* = M^*$. A path from x to y in $\mathcal{G}(M)$ achieving the minimal cost among the paths of any length is called a *geodesic* joining x to y. We will still call a geodesic an optimal path when the matrix M is nonsymmetric.

4. MIN-PLUS CLOSED JACKSON SERVICE NETWORKS

A closed Jackson network of queues is a set of n customers and m services. The customers wait for services in queues attached to each service. The customers are served in the order of arrival. The service is random and markovian. In discrete time situation, a (m, m) transition probability matrix r is given. The entry r_{ij} is the probability that a customer, served at queue i, goes to queue j, if the queue i is not empty. If the queue is empty, this probability is 0. Such a system is a Markov chain with state space :

$$S^m_n \stackrel{\text{def}}{=} \{x \in \mathbb{N}^m : 1.x = n\}\ ,$$

where 1 denotes the vectors with all its entries equal to 1 with size adapted to the context (here m). It is clear that, if r is irreducible, the Markov chain describing the system is irreducible. Therefore it has a unique invariant measure p. This measure is explicitly computable :

$$p_x = k\theta_1^{x_1} \cdots \theta_m^{x_m}\ ,$$

with θ any solution of $\theta r = \theta$ and k a normalizing constant such that $p1 = 1$.

The best way to understand what is a min-plus closed Jackson service network is to consider the following problem. We consider a company renting cars. It has n cars and m parkings in which customers can rent cars. The customers can rent a car in a parking and leave the rented car in another parking. After some time the distribution of the cars in the parkings is not satisfactory and the company has to transport the cars to achieve a better distribution. Given r the (m, m) matrix of transportation cost from a parking to another, the problem is to determine the minimal cost of the transportation from a distribution $x = (x_1, \cdots, x_m)$ of the cars in the parking to another one $y = (y_1, \cdots, y_m)$ and to compute the best plan of transportation. Therefore the precise transportation problem is the following.

MIN-PLUS CLOSED JACKSON PROBLEM (TRANSPORTATION PROBLEM). *Given the (m, m) transition cost matrix r irreducible such that $r_{ij} > 0$ if $i \neq j = 1, \cdots, m$ and $r_{ii} = 0$ for all $i = 1, \cdots, m$, compute M^* for the the Bellman chain on S_n^m of transition cost M defined by $M_{x,T_{ij}(x)} \stackrel{\text{def}}{=} r_{ij}$ and*

$$T_{ij}(x_1, \cdots, x_m) \stackrel{\text{def}}{=} (x_1, \cdots, x_i - 1, \cdots, x_j + 1, \cdots, x_m) ,$$

for $i, j = 1, \cdots, m$.

The operator T_{ij} corresponds to the transportation of a car from the parking i to the parking j. We denote $\mathcal{T} \stackrel{\text{def}}{=} \{T_{ij},\ i, j = 1, \cdots, m\}$.

If $r_{ii} = e$ for all $i = 1, \cdots, m$ (the absence of transportation costs nothing) the previous problem corresponds to the computation of the largest invariant cost c satisfying $c = cM$, and $c_x = e$. Indeed, in this case the left eigen semimodule has as many independent generators as states[1]. Remarking that the diagonal entries of M^* are e, it is clear $M^*_{x.} M = M^*_{x.}$. Then, from the fact that $q = bM^*$ is the largest solution of $q = qM \oplus b$, we can prove that the searched extremal left eigenvector is $M^*_{x.}$.

5. SOLUTION OF THE 2-PARKINGS TRANSPORTATION PROBLEM

This transportation problem is trivial in the 2-parkings case. Let us denote : $a \stackrel{\text{def}}{=} r_{12}$, $b \stackrel{\text{def}}{=} r_{21}$ and x the number of cars in the first parking called A. The number of cars in the second parking B is $n - x$. Therefore a possible state of the system is x. The transition cost matrix M is :

$$M = \begin{pmatrix} e & b & \epsilon & \cdot & \epsilon \\ a & e & b & \cdot & \epsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \epsilon & \cdot & a & e & b \\ \epsilon & \cdot & \epsilon & a & e \end{pmatrix}$$

The computation of M^* is easy in this case :

$$M^* = \begin{pmatrix} e & b & b^2 & \cdot & b^n \\ a & e & b & \cdot & b^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a^{n-1} & \cdot & a & e & b \\ a^n & \cdot & a^2 & a & e \end{pmatrix} .$$

Suppose that $x > y$, the entry $M^*_{xy} = a^{x-y}$ in the min-plus algebra corresponds to $a(x - y)$ in the standard algebra which is transportation cost of $x - y$ cars from A to B. Similarly, if $x < y$, $M^*_{xy} = b^{y-x}$ corresponds to the transportation cost $y - x$ cars from B to A.

In the general case $m > 2$ it is not easy to build and manipulate matrix M but the 2-parkings case suggests that simple formulae exist.

[1] Let us recall that in the min-plus context the irreducibility of the transition matrix assures the uniqueness of the eigenvalue but not the uniqueness of the generators of the eigen semimodule see [6] Section 3.7.

6. SOLUTION OF THE m-PARKINGS TRANSPORTATION PROBLEM

Let us consider the m-parkings case. In this case a path $\pi \in \mathcal{P}$ is

$$xT^1(x)T^2 \circ T^1(x) \cdots y = T^l \circ T^{l-1} \circ \cdots T^1(x) ,$$

with $T^i \in \mathcal{T}$. Since the arcs $\mathcal{P}_r$ of r are $xT_{ij}(x)$ with $x \in \mathcal{N}$ and $T_{ij} \in \mathcal{T}$ we can code ($\simeq$) a path $\pi \in \mathcal{P}^*$ in a simpler way by the couple $\pi \simeq x\mu$ with $x \in \mathcal{N}$ a node of $\mathcal{G}(M)$ and $\mu \in \mathcal{P}_r^*$ a path of $\mathcal{G}(r)$. Clearly we have :

$$\pi(M) = \mu(r), \ \forall \pi \simeq x\mu \in \mathcal{P}^* .$$

Remarking that the vector $T_{ij}(x) - x$ is independent of x let us call it γ_{ij} and denote $\Gamma = \{\gamma_{ij}, \ i, j = 1, \cdots, m\}$. These vectors are not mutually independent indeed we have the relations :

$$\gamma_{ik} = \gamma_{ij} + \gamma_{jk}, \ \forall i, j, k = 1, \cdots, m .$$

For a path $\mu \in \mathcal{P}_r^*$, the evaluation $\mu(\Gamma) \in \mathbb{Z}^m$ is obtained by using the morphism which to the concatenation associates the vectorial sum and to the letters associate the corresponding vectors of Γ. For example for the path $ijkl \in \mathcal{P}_r^*$ we have :

$$ijkl(\Gamma) = \gamma_{ij} + \gamma_{jk} + \gamma_{kl} .$$

Then, the constraint on the paths $\pi : \langle \pi \rangle = xy$ with $x, y \in \mathbb{Z}^m$ is equivalent to the constraint $\mu(\Gamma) = y - x$ for the path $\pi \simeq x\mu$.

The cost of a path $\mu(r)$ depends only of the number of times each arc appears in μ and not of the order of the arcs. Similarly the constraint $\mu(\Gamma) = y - x$ does not depend of the order of the arcs in the path μ, since the evaluation $\mu(\Gamma)$ corresponds to additions of vectors, and addition of vector is commutative. To take account of this symmetry of the problem we denote $\mathcal{P}_r^c$ the set of equivalent classes of paths (where two paths are equivalent if the arcs appear the same number of times). Therefore, for $\mu \in \mathcal{P}_r^c$ we can take the representative $\mu = \prod_{a \in \mathcal{P}_r} a^{n_a}$. For example the path $\mu = ijijk$ belongs to the class of $(ij)^2(ji)(jk)$.

It is clear that $\mu(r^*) \leq \mu(r)$ because $r_{ij} \geq r_{ij}^*$. Moreover for each μ it exists $\tilde{\mu}$ such that $\mu(r^*) = \tilde{\mu}(r)$ and $\mu(\Gamma) = \tilde{\mu}(\Gamma)$. The path $\tilde{\mu}$ is obtained by substituting the arcs ij of μ by paths μ_{ij} such that $\mu_{ij}(r_{ij}) = r_{ij}^*$. Inside S_n^m this substitution is always possible. This is not always possible on the boundary of S_n^m because the path $x\mu$ may leave S_n^m. To avoid this difficulty we suppose that the costs on the boundary arcs are not r_{ij} but r_{ij}^*.

We can summarize the previous considerations in the following proposition.

PROPOSITION 3. *The optimal value of the transportation problem is :*

$$M_{xy}^* = \mathcal{P}_{xy}^*(M) = \Phi_{r^*}(y - x) ,$$

with

$$\Phi_{r^*}(z) \stackrel{\text{def}}{=} \bigoplus_{\substack{\mu \in \mathcal{P}_{r^*}^c \\ \mu(\Gamma) = z}} \mu(r^*) .$$

The mathematical program $\Phi_{r^*}(z)$ is a flow problem.

PROPOSITION 4. *Denoting by $\mathcal{J}$ the incidence matrix nodes-arcs of the complete graph with m nodes we have :*

$$\Phi_{r^*}(z) = \inf_{\substack{\phi \geq 0 \\ \mathcal{J}\phi = z}} \phi . r^* ,$$

where $\phi.r = \sum_{i,j} r_{ij}\phi_{ij}$.

Proof. If we denote by ϕ_{ij} the exponent of the arc ij in the word $\mu \in \mathcal{P}^c_{r^*}$ the criteria of $\Phi_{r^*}(z)$ gives $\phi.r^*$, its constraints are $\mathcal{J}\phi = z$. The only point to verify is that : to the path $\mu = \prod (ij)^{\phi_{ij}}$, associated to the optimal ϕ, corresponds a path in the class of $\pi \simeq x\mu$ whose visited nodes belong to S^m_n. Let us suppose that it is not the case, it would exist another node t on the optimal path π and a coordinate $i \in \{1, \cdots, m\}$ such that $t_i = \inf(x_i, y_i)$. Indeed, if π leave S^m_n somewhere, it is necessary that a coordinate of one of its nodes becomes negative. Let us suppose that $t_i = x_i$ (the arguments are the same in the other case). The geodesic $x\pi_1 \cdots \pi_k t$ from x to t would satisfy :

$$\pi_{l,i} = x_i, \ l = 1, \cdots, k . \tag{2}$$

Indeed, if we consider the flow problem associate to this new geodesic problem it would have the constraint $\mathcal{J}\phi = z$ with $z_i = 0$ and the optimal flow would satisfy $\phi_{ik} = \phi_{li} = 0$ for all k and l because, at node i, there is neither production nor consummation and the transport cost on lk is $r^*_{lk} \leq r^*_{li} + r^*_{ik}$ by definition of r^*. This implies that we can reduce this “x to t geodesic” problem to a transportation problem without the parking i. All the paths, associated to this reduced flow problem, satisfy (2). This argument shows that it exists a path associated to the optimal flow, such that, for all node l and component i, $\pi_{l,i} \geq \inf(x_i, y_i)$ which is a contradiction with the fact that all the these paths are supposed to leave S^m_n. □

COROLLARY 5. *We have for all y and x satisfying $x_j = 0$ for $j \neq i$ and $x_i = n$*

$$M^*_{xy} = \bigotimes_{j, j\neq i} (r^*_{ij})^{y_j} ,$$

and for all x and y satisfying $y_j = 0$ for $j \neq i$ and $y_i = n$

$$M^*_{xy} = \bigotimes_{j, j\neq i} (r^*_{ji})^{x_j} .$$

Proof. In these two cases the flow problems are trivial. The nonnul components are respectively $\phi_{ij} = y_j$ and $\phi_{ji} = x_i$, for $j \neq i$. □

This corollary gives the searched min-plus product form. In the future we will try to extend this result, as it has been done in probability to more general problem, for example when the transition costs depend of the number of cars in the parkings.

7. EXAMPLE

Let us consider the transportation system with 3 parkings and 6 cars, and transportation costs :

$$r = \begin{pmatrix} 0 & 1 & +\infty \\ +\infty & 0 & 1 \\ 1 & +\infty & 0 \end{pmatrix} = \begin{pmatrix} e & 1 & \epsilon \\ \epsilon & e & 1 \\ 1 & \epsilon & e \end{pmatrix} .$$

We have :

$$r^* = \begin{pmatrix} e & 1 & 2 \\ 2 & e & 1 \\ 1 & 2 & e \end{pmatrix} .$$

Let us suppose that $x = (0, 0, 6)$ and $y = (2, 3, 1)$, we can apply the corollary, we have :

$$M^*_{xy} = (r^*_{31})^2 (r^*_{32})^3 = 2 \times 1 + 3 \times 2 = 8 \; .$$

The Geodesic is given in Fig.1.

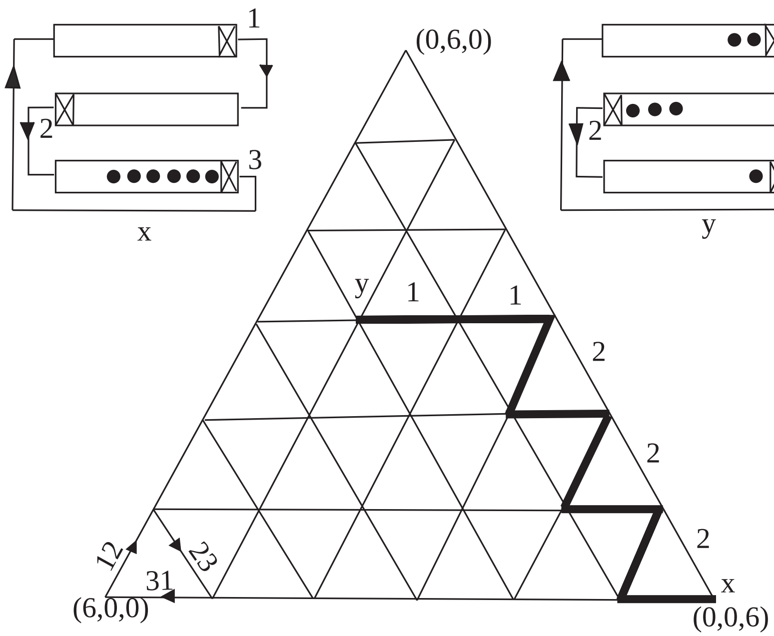

FIGURE 1. Transportation System (6 cars, 3 parking).

REFERENCES

[1] M. Akian : "Densities of idempotent measures and large deviations", to appear AMS, and INRIA Report N. 2534 (1995).

[2] M. Akian : "Theory of cost measures: convergence of decision variables", INRIA Report N. 2611 (1996).

[3] M. Akian, J.P. Quadrat and M. Viot : "Bellman Processes" L.N. in control and Inf. Sciences N. 199, Ed. G. Cohen, J.P. Quadrat, Springer Verlag (1994).

[4] M. Akian, J.P. Quadrat and M. Viot : "Duality between probability and optimization" In J. Gunawardena (Editor), "Idempotency", Cambridge University Press (1998).

[5] F. Baskett, F. Chandy, M. Muntz and R. Palacios : "Open, closed and mixed networks of queues with different classes of customers", JACM 22 (1975).

[6] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat : "Synchronization and Linearity", Wiley (1992).

[7] F. Bellalouna : "Un point de vue linéaire sur la programmation dynamique. Détection de ruptures dans le cadre des problèmes de fiabilité", Thesis dissertation, University of Paris-IX Dauphine (1992).

[8] D. Bertsimas and J. N. Tsitsiklis : "Linear Optimization", Athena Scientific, Belmont Mass. (1997).

[9] P. J. Courtois : "Decomposability Queuing and Computer System Applications", Academic Press, New York (1977).

[10] N. M. Van Dijk : "Queuing networks and Product forms : a system approach", Wiley (1993).

[11] M. Gondran and M. Minoux : "Graphs and Algorithms", Wiley, (1984).

[12] P. Del Moral : "Résolution particulaire des problèmes d'estimation et d'optimisation non-linéaires", Thesis dissertation, Toulouse, France (1994).

[13] P. Del Moral, T. Thuillet, G. Rigal and G. Salut : “Optimal versus random processes : the nonlinear case”, rapport de recherche LAAS, (1990)

[14] G. Fayolle and J.M. Lasgouttes : “Asymptotics and scalings for Large Product-Form Networks via the Central Limit Theorem”, Markov Processes and Related Fields, V.2, N.2, p.317–349 (1996).

[15] S. Gaubert : “Théorie des systèmes linéaires dans les dioides”, Thesis dissertation, École des Mines de Paris, (1992).

[16] J. Gunawardena (Editor) : “Idempotency”, Cambridge University Press (1998).

[17] J.R. Jackson “Networks of waiting lines”, Operations Research 5, 518-521.

[18] S. Gaubert and M. Plus : “Methods and applications of (max,+) linear algebra”, in R. Reischuk and M. Morvan, editors, *STACS’97*, number 1200 in LNCS, Lübeck, Springer, (1997).

[19] F. P. Kelly : “Reversibility and Stochastic Networks”, Wiley, (1979).

[20] J. G. Kemeny and J. L. Snell : “Finite Markov Chains”, Van Nostrand, Princeton, N.J. (1967).

[21] V. N. Kolokoltsov and V. P. Maslov : “Idempotent Analysis and Its Applications”, Kluwer (1997).

[22] V.P. Maslov and S.N. Samborskii : “Idempotent Analysis”, AMS (1992).

[23] V.P. Maslov : “Méthodes Opératorielles”, Éditions MIR, Moscou (1987).

[24] E. Pap : “Null-Additive Set Functions”, Mathematics and Its applications 337, Kluwer academic Publishers, Dordrecht (1995).

[25] J.P. Quadrat : “Théorèmes asymptotiques en programmation dynamique”, Note CRAS 311, p.745-748, Paris (1990).

[26] P. Whittle : ‘’ Systems in Stochastics Equilibrium”, Wiley, New York (1986).