

# About Dynamical Systems Appearing in the Microscopic Traffic Modeling

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## Abstract

Motivated by microscopic traffic modeling, we analyze dynamical systems which have a piecewise linear concave dynamics not necessarily monotonic. We introduce a deterministic Petri net extension where edges may have negative weights. The dynamics of these Petri nets are well-defined and may be described by a generalized matrix with a submatrix in the standard algebra with possibly negative entries, and another submatrix in the minplus algebra. When the dynamics is additively homogeneous, a generalized additive eigenvalue may be introduced, and the ergodic theory may be used to define a growth rate under additional technical assumptions. In the traffic example of two roads with one junction, we compute explicitly the eigenvalue and we show, by numerical simulations, that these two quantities (the additive eigenvalue and the growth rate) are not equal, but are close to each other. With this result, we are able to extend the well-studied notion of fundamental traffic diagram (the average flow as a function of the car density on a road) to the case of roads with a junction and give a very simple analytic approximation of this diagram where four phases appear with clear traffic interpretations. Simulations show that the fundamental diagram shape obtained is also valid for systems with many junctions. To simulate these systems, we have to compute their dynamics, which are not quite simple. For building them in a modular way, we introduce generalized parallel, series and feedback compositions of piecewise linear concave dynamics.

## I. INTRODUCTION

The main purpose of this paper is to explore some new points on dynamical systems that have emerged with the study of microscopic traffic modeling using minplus algebra and Petri nets. These points are the following :

- Standard Petri nets or linear maxplus algebra have not enough modeling power to describe the dynamics of a traffic system as simple as two roads with a crossing and the simplest vehicle move.

- A remedy to this difficulty is to introduce possibly negative weights on Petri net arcs (not to be confused with negative places or negative tokens as in [41]) or to use nonlinear minplus dynamics that are not monotonic.
- A consequence is that all known standard methodology based mainly on dynamic programming is ineffective.
- Nevertheless, it is sometimes possible to use the standard ergodic theory, and/or to compute a generalized eigenvalue for the subclass of additively homogeneous systems and get nice qualitative results on the system.
- When it is not possible to derive analytical results, we can do numerical simulations. In general, a microscopic modeling of a realistic system has a very large number of state variables difficult to manage. However, for the construction of the model, a generalized matrix calculus using minplus algebra can be developed for the class of system described here (having piecewise affine concave dynamics).

All these points are illustrated on the traffic application for which we have obtained a new result: a good analytic approximation of the macroscopic law called the fundamental diagram in the case of roads with junctions.

The traffic on a road has been studied from different points of view on a macroscopic level. For example:

- The Lighthill-Whitham-Richards Model [34], which is the most standard view, expresses the mass conservation of cars seen as a fluid:

$$\begin{cases} \partial_t \rho + \partial_x \varphi = 0, \\ \varphi = f(\rho), \end{cases}$$

where  $\varphi(x, t)$  denotes the flow at time  $t$  and position  $x$  on the road;  $\rho(x, t)$  denotes the density; and  $f$  is a given function called the *fundamental traffic diagram*. For traffic, this diagram plays the role of the perfect gas law for the fluid dynamics. The diagram has been estimated using experimental data, and its behavior is quite different from standard gas at high density. We obtain an idea of the diagram's shape in the subsection titled "traffic on a circular road."

Daganzo in [13] using the variable  $N(t, x)$  counting the cumulated number of vehicles having reached the point  $x$  at time  $t$ , and remarking that  $\varphi = \partial_t N$  and  $\rho = -\partial_x N$  has

interpreted the equation  $\varphi = f(\rho)$  as an Hamilton-Jacobi equation when  $f$  is concave. The other equation  $\partial_t \rho + \partial_x \varphi = 0$  telling that  $\partial_{tx} N = \partial_{xt} N$ . Then, in [14] he had generalized to a network by postulating the existence of an Hamiltonian in this case also and by trying to approximate it as an infimum of linear constraints that he obtains by physical considerations on the street involved. In [24], [25] he has studied what happen experimentally on real towns.

- The kinetic model (Prigogine-Herman [44]) gives the evolution of the density of particles  $\rho(t, x, v)$  as a function of  $t, x$  and  $v$ , where  $v$  is the speed of particles. The model is given by:

$$\partial_t \rho + v \partial_x \rho = C(\rho, \rho) ,$$

where  $C(\rho, \rho)$  is an interacting term that is, in general, quadratic in  $\rho$  and, as such, models the driver behavior in a simple way. From the distribution  $\rho$ , we can derive all the useful quantities, such as the average speed  $\bar{v}(t, x) = \int v \rho(t, x, v) dv$ .

The integration of the second model is more time consuming and, therefore, not used in practice. The main interest of the Prigogine-Herman equation is that we can derive macroscopic laws like the fundamental diagram from its solution.

The first model assumes the knowledge of the fundamental traffic diagram  $f$ . This function has been studied not only experimentally but also theoretically using simple microscopic models such as exclusion processes, [16], [7], cellular automata [3], [9], or simulation of individual car dynamics. Here, we recall a way to derive an approximation of this diagram. This derivation consists of computing the eigenvalue of a simple minplus linear system counting the number of vehicles entering in a road section.

The main result obtained is the generalization of this fundamental law to the 2D case where the roads cross each other. In statistical physics, a lot of numerical work ([40], [12], [39], [5], [8]) and good surveys [9], [26] have been done on idealized towns. These works analyze numerical experimentations based on various stochastic models with or without turning possibilities and show the existence of a threshold of the density at which the system blocks suddenly. The particular case of one junction is studied in [28], [29] where precise results are given for the stochastic case without turning possibilities. Let us mention also the attempt to derive a 2D fundamental diagram by Helbing [27] using queuing theory. In the saturated and unsaturated traffic case he derives a formula for an intersection that he extends to an area of a town that fits

well to experimental datas.

Here we present a deterministic model with turning possibilities, based on Petri nets and minplus algebra. The minplus linear model on a unique road can be described in terms of event graph which is a subclass of Petri nets. The presence of conflicts at junctions prevents the extension of this model to the 2D case.

We propose a way to solve the difficulty by extending the class of weights used in Petri nets allowing negative weights. Due to such weights, the firing of a transition can consume tokens downstream, and the modeling power of the Petri nets is improved significantly. This possibility is used to model the authorization to enter in the junctions.

The dynamics of general Petri nets allowing negative weights can be written easily but is neither linear in minplus algebra nor monotone. Nevertheless, in the traffic applications given here, dynamics ( $x^{k+1} = f(x^k)$ ) are always homogeneous of degree one ( $f(\lambda \otimes x) = \lambda \otimes f(x)$ , where  $\otimes$  denotes the minplus multiplication that is the standard addition). For such systems, the eigenvalue problem (computation of  $\lambda$  such that  $\lambda \otimes x = f(x)$ ) can be reduced to a fixed point problem. The existence of the growth rate  $\chi = \lim_k x^k/k$  is due to the existence of a Birkhoff average. The quantities  $\lambda$  and  $\chi$  coincide when  $f$  is also monotone, but this is not true in the general case. The monotone case has been studied carefully in [32], [38]. In traffic examples of roads with junctions, the dynamics is not monotone.

In all the traffic examples, we study systems of roads on a torus without entries in such a way that the number of vehicles remains constant in the system. In this way, we represent an idealization of constant densities for a more realistic system. To maintain a constant density in an open system, a new vehicle has to enter each time another one leaves the system. This is mathematically equivalent to consideration of circular roads. If we want to maintain a constant local density in a large system, it is the same problem; each time a vehicle leaves the local subsystem, another one has to enter.

The particular case of two circular roads with one junction is studied in detail. For this system, a result on the existence of the growth rate is obtained using the nondecreasing trajectory property of the states starting from zero. From this result, we show that the distances between the states stay bounded. The eigenvalue problem can be solved explicitly. On simulation, we see that the eigenvalue and the growth rate generally do not coincide. However, these quantities are very close for any fixed density and any fixed ratio. Therefore, the simple formulas obtained for the

eigenvalue give a good approximation of the traffic fundamental diagram.

This definition of the fundamental diagram must not be confused with the one used by Daganzo which is the Hamiltonian of the traffic dynamics interpreted as a dynamic programming equation in the simplest case of a single road and generalized in heuristic way to the city case. His fundamental diagram mainly coincides with our minplus dynamics, in the case of one street, but is observed experimentally not derived of a microscopic model as we do here. The dependence of the eigenvalue or the average growth rate with the density of vehicles in the system, which is a difficult result to obtain even in the case of two streets with one intersection, is not done in the Daganzo work. Moreover, for the city case, we remark also, on the simple example studied here, that the dynamics is not a standard dynamic programming equation associated to a deterministic or a stochastic control problem (since the dynamics is not monotone) which could be a problem, at least at the conceptual level, for the point of view adopted by Daganzo.

The fundamental diagram shows four phases:

- *Free phase*: The density is low where the vehicles do not interact.
- *Saturation phase*: The junction is saturated, but the locations downstream of the junction are free when a vehicle wants to leave the junction.
- *Recession phase*: The locations downstream of the junction are sometimes crowded when a vehicle wants to leave the junction.
- *Freeze phase*: The vehicles cannot move.

These phases are derived from a unique simple model and are not postulated to obtain different models subsequently analyzed.

Preliminary results on the traffic dynamics have been presented in [18], [19], [20]. In [21], many developments, complementary results, and other completely solved examples can be found. The theorems given here on the eigenvalue problem and the growth rate complete some of the main results of [21] by relaxing some assumptions and clarifying the growth rate existence problem.

We show that a system theory can be developed for the class of concave polyhedral 1-homogeneous systems, which we construct using parallel, series, and feedback compositions. Moreover, the dynamics of these systems are characterized by the composition of a standard matrix and a minplus matrix. The sum of the entries in each line of the standard matrix is equal to one. We can compute easily the transformation on these two matrices corresponding to the

three composition operations. The interest of this “system theory” can be shown by building the traffic dynamics of a regular town starting from three elementary systems. We do not detail the construction, but we show a stationary vehicle distribution obtained for a simple regular town with dynamics built in that way. In this case, the fundamental diagrams still present the four phases observed in the simple case of two roads with one junction. All the numerical simulations have been done using the ScicosLab software [46].

This paper is divided into three parts. First, we recall the basic results of minplus algebra; we present a system theory for polyhedral concave 1-homogeneous systems and discuss their growth rate and eigenvalue problems. Second, we present Petri nets allowing negative weights. Third, we give applications to the computation of the fundamental traffic diagram; we consider the case of one circular road with minplus linear dynamics and study the case of two circular roads with a junction in detail; then, we explain briefly the way to build the dynamics using the system theory described in the first part.

## II. MINPLUS ALGEBRA AND EXTENSIONS

### A. Review of minplus algebra

In this section, we revisit the main definitions and properties of the minplus algebra. An in-depth treatment of the subject is in [4].

The structure  $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$  is defined by the set  $\mathbb{R} \cup \{+\infty\}$  endowed with the operations  $\min$  (denoted by  $\oplus$ , called *minplus sum*) and  $+$  (denoted by  $\otimes$ , called *minplus product*). The element  $\varepsilon = +\infty$  is the *zero* element  $\varepsilon \oplus x = x$ , and is *absorbent*  $\varepsilon \otimes x = \varepsilon$ . The element  $e = 0$  is the *unit* element  $e \otimes x = x$ . The main difference with respect to the conventional algebra is the idempotency of the addition  $x \oplus x = x$  and the fact that the addition cannot be *simplified*  $a \oplus b = c \oplus b \not\Rightarrow a = c$ . This structure is called *minplus algebra*. We will call  $\overline{\mathbb{R}}_{\min}$  the completion of  $\mathbb{R}_{\min}$  by  $-\infty$  with  $-\infty \otimes \varepsilon = \varepsilon$ .

This minplus structure on scalars induces an idempotent semiring structure on  $m \times m$  square matrices with the element-wise minimum denoted by  $\oplus$  and the matrix product defined by  $(A \otimes B)_{ik} = \min_j (A_{ij} + B_{jk})$ , where the zero and the unit matrices are still denoted by  $\varepsilon$  and  $e$ . We associate a precedence graph  $\mathcal{G}(A)$  to a square matrix  $A$  where the nodes of the graph correspond to the columns of the matrix  $A$  and the edges of the graph correspond to the non-zero ( $\neq \varepsilon$ ) entries of the matrix. The *weight* of the edge going from  $i$  to  $j$  is the non-zero entry  $A_{ji}$ .

We define the weight of a path  $p$  in  $\mathcal{G}(A)$ , which we denote by  $|p|_w$ , as the minplus product of the weights of the edges composing the path (that is the standard sum of weights). The number of edges of a path  $p$  is denoted by  $|p|_l$ . We will use the following fundamental result (see [4]).

*Theorem 1:* If the graph  $\mathcal{G}(A)$  associated with the  $m \times m$  minplus matrix  $A$  is strongly connected, then the matrix  $A$  admits a unique *eigenvalue*  $\lambda \in \mathbb{R}_{\min} \setminus \{\varepsilon\}$ :

$$\exists X \in \mathbb{R}_{\min}^m, X \neq \varepsilon : A \otimes X = \lambda \otimes X \quad \text{with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l}, \quad (1)$$

where  $\mathcal{C}$  is the set of circuits of  $\mathcal{G}(A)$   $\square$

### B. A generalized matrix calculus

In a Petri net, two kinds of operations appear. One is the accumulation of resources in the places, and the other is the synchronization in the transitions. The first operation can be modeled by addition, and the second by a min or max (a task can start at the maximum of the arrival instants of the resources needed by the task). Matrix notations can be generalized to such situations.

We consider the set of  $m \times m$  matrices where the rows [resp. columns] are partitioned in two sets the standard and the minplus (here the  $m'$  first rows [resp. columns] and the last  $m''$  rows [resp. columns]) with entries in  $\mathbb{R} \cup \{+\infty, -\infty\}$ , equipped with the two operations  $\boxplus$  and  $\boxtimes$  defined by:

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxplus \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} &= \begin{bmatrix} A + A' & B + B' \\ C \oplus C' & D \oplus D' \end{bmatrix}, \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxtimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} &= \begin{bmatrix} AA' + BC' & AB' + BD' \\ C \otimes A' \oplus D \otimes C' & C \otimes B' \oplus D \otimes D' \end{bmatrix}. \end{aligned}$$

Since the entries are in an extension of  $\mathbb{R}$ , we have to specify the scalar addition and multiplication table:

$$0 \times \pm\infty = \pm\infty \times 0 = 0, \quad +\infty \otimes (-\infty) = +\infty - \infty = +\infty.$$

These choices have been made to preserve the absorption properties for the multiplication of the null elements of the standard algebra (0) and the minplus algebra ( $\varepsilon = +\infty$ ). This absorption property is useful to model the absence of the arc in the precedence graph  $\mathcal{G}(A)$  associated to a square matrix  $A$  (defined in the same way as in the pure minplus case).

The addition  $\boxplus$  is associative, commutative and has the null element  $\begin{bmatrix} 0 & 0 \\ \varepsilon & \varepsilon \end{bmatrix}$  still denoted  $\varepsilon$ .

The multiplication  $\boxtimes$  has no identity element. It is neither associative nor commutative nor distributive with respect to the addition.

The main interest of this operation appears when the left operand is a vector  $Y = A \boxtimes X$  where  $X \in \overline{\mathbb{R}}_{\min}^m$  is a vector and  $A$  is a  $m \times m$  matrix with entries in  $\overline{\mathbb{R}}_{\min}$ , then  $Y$ , seen as a function of  $X$ , is a set of  $m'$  standard linear forms and of  $m''$  minplus linear forms on  $\overline{\mathbb{R}}_{\min}^m$  with  $m = m' + m''$ . The operation  $Z = A \boxtimes (A \boxtimes X)$  corresponds to the compositions of these linear forms but these compositions do not define anymore a set of standard and minplus linear forms and  $Z \neq (A \boxtimes A) \boxtimes X$ .

Moreover, when the left operand is a vector, this multiplication can be represented by the graph  $\mathcal{G}(A)$  where we have two kinds of nodes (those corresponding to the standard  $+$  operation, and those corresponding to  $\oplus$  operation) and two kinds of edges (those which operate multiplicatively ( $\times$ ) and those which operate additively ( $\otimes$ )).

*Example 1:* Let us consider the graph  $\mathcal{G}(A)$  associated to the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  (Figure-1) with one node associated to the standard algebra and one node to the minplus algebra. Then  $y = A \boxtimes x$ , where  $y$  and  $x$  are two vectors with two entries, means:

$$y_1 = ax_1 + bx_2, \quad y_2 = \min(c + x_1, d + x_2).$$

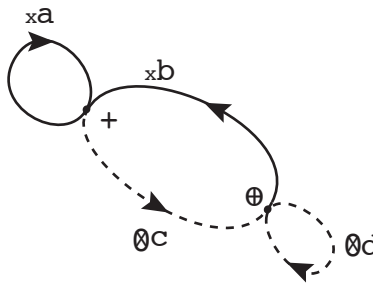


Fig. 1. The incidence graph associated to the matrix  $A$  with one  $\oplus$ node and one  $+$ node.

We adopt the following conventions to solve some ambiguities in the formula notations:



– As soon as a minplus symbol appears in a set of formulas, all the operations must be understood in the minplus sense with the exception of the exponent that must be understood in the standard sense. For example,  $x^{a/b} \oplus 1$  or  $\sqrt[b]{x^a} \oplus 1$  must be understood as  $\min((a/b)x, 1)$  and not as  $\min((a - b)x, 1)$ . In minplus sense,  $a/b = a - b$  since it is the solution<sup>1</sup> of  $b \otimes x = a$ . The equation  $b \otimes x = a$  means  $b + x = a$ .

– In a minplus formula, the rational numbers (written with figures) are denoted in the standard algebra. For example,  $\frac{1}{2}x \oplus 1$  (instead of  $\sqrt{1}x \oplus 1$ ) means  $\min(0.5 + x, 1)$  and not  $\min(-1 + x, 1)$ , but  $(a/b)x \oplus 1$  means  $\min(a - b + x, 1)$ .

– A non-zero element in the minplus sense means a finite element in the usual sense. Positive element will be used always in the standard sense.

With these conventions the formulas are concise and not ambiguous.

### C. A generalized system theory

We can develop a generalized system theory based on the two operations  $\boxplus$  and  $\boxtimes$ . We partition the states [resp. inputs, outputs] into two classes, the standard states [resp. inputs, outputs] and the minplus states [resp. inputs, outputs]. Then the dynamics can be written:

$$\begin{bmatrix} X^{k+1} \\ Y^{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & \varepsilon \end{bmatrix} \boxtimes \begin{bmatrix} X^k \\ U^k \end{bmatrix}$$

These dynamics, denoted by  $S$  and defined by the matrices  $(A, B, C)$ , associate the output signals  $(Y^k)_{k \in \mathbb{N}}$  to the input signals  $(U^k)_{k \in \mathbb{N}}$ . We write  $Y = S(U)$ .

Let us note that  $\boxtimes$  corresponds to the definition given previously when there is a permutation of rows and columns. This is because  $[X^k \ U^k]'$  [resp.  $[X^k \ Y^k]'$ ] is not written in the canonical form since all the standard entries are not followed by all the minplus entries but by the standard states, the minplus states, the standard inputs [resp. outputs], and the minplus inputs [resp. outputs]. See the Petri net application (6) for developed formulas.

On these systems, we define the following operations:

<sup>1</sup>For the reader familiar with the residuation, the minplus division used here means the standard minus operator with the convention previously given for infinite elements. This choice is incompatible with the residuation which chooses the smallest solution of  $b \otimes x = a$ .

- **Parallel Composition.** Given two systems  $S_1$  and  $S_2$  (with the same numbers of inputs and outputs), we define the system  $S = S_1 \boxplus S_2$  obtained by using the same entries and adding<sup>2</sup> the outputs. The dynamics of  $S$  is defined by:

$$A = \begin{bmatrix} A_1 & \varepsilon \\ \varepsilon & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \ C_2].$$

- **Series Composition.** Given two systems  $S_1$  and  $S_2$  (where the output numbers of  $S_2$  equal the input numbers of  $S_1$ ), we define the system  $S = S_1 \boxtimes S_2$  obtained by composition of the two systems:  $S(U) = S_1(S_2(U))$ . Using the new state  $[X_1 \ X_2 \ Y_2]'$ , the dynamics of  $S$  is defined by:

$$A = \begin{bmatrix} A_1 & \varepsilon & B_1 \\ \varepsilon & A_2 & \varepsilon \\ \varepsilon & C_2 & \varepsilon \end{bmatrix}, B = \begin{bmatrix} \varepsilon \\ B_2 \\ \varepsilon \end{bmatrix}, C = [C_1 \ \varepsilon \ \varepsilon].$$

- **Feedback.** Given a system  $S$ , we define the feedback system  $S^\square$  as the solution in  $Y$  of  $Y = S(U \boxplus Y)$ . Using the new state  $[X \ Y]'$ , the dynamics of  $S^\square$  are defined by:

$$A^\square = \begin{bmatrix} A & B \\ C & \varepsilon \end{bmatrix}, B^\square = \begin{bmatrix} B \\ \varepsilon \end{bmatrix}, C^\square = [C \ \varepsilon].$$

#### D. Additively homogeneous autonomous dynamic systems

Let us now discuss one-homogeneous minplus dynamical systems which are a large class applied frequently when there is a conservation such as probability mass or tokens in Petri nets. However, because we accept negative entries, we obtain a generalization of two cases. One case is a measure not necessarily positive with a total mass equal to one. The other case is a conservation of tokens in a situation where negative entries may appear. Let us start with an academic example. The traffic modeling will give more concrete examples in the following sections.

*Example 2:* Let us go back to Example 1 and assuming that  $a + b = 1$ . Adding  $\lambda$  to each component of  $x$  (that we can write  $\lambda \otimes x$ ) implies that the two components of  $y$  are augmented by  $\lambda$ . We have

$$A \boxtimes (\lambda \otimes x) = \lambda \otimes (A \boxtimes x).$$

<sup>2</sup>The standard output nodes are added in the standard algebra and the minplus output nodes are added in the minplus algebra.

We say that the system is *additively homogeneous of degree 1* or, more simply, *homogeneous*. Indeed, using the minplus notation,  $y = A \boxtimes x$  can be written:

$$y_1 = (x_1)^{\otimes a} \otimes (x_2)^{\otimes b}, \quad y_2 = c \otimes x_1 \oplus d \otimes x_2,$$

which is clearly homogeneous of degree 1 in the minplus algebra as long as  $a + b = 1$ . Moreover, if  $a$  and  $b$  are nonnegative, then the transformation is *nondecreasing*.

To simplify the notations, we will write the transformation in the following way:

$$y_1 = (x_1)^a (x_2)^b, \quad y_2 = cx_1 \oplus dx_2.$$

More generally, we say that the function  $f : \overline{\mathbb{R}}_{\min}^n \mapsto \overline{\mathbb{R}}_{\min}^n$  is *homogeneous* if

$$f(\lambda \otimes x) = \lambda \otimes f(x).$$

### E. Eigenvalues of homogeneous systems

The *eigenvalue problem* of such a function can be formulated as finding non-zero  $x \in \overline{\mathbb{R}}_{\min}^n$  and  $\lambda \in \overline{\mathbb{R}}_{\min}$  such that:

$$\lambda \otimes x = f(x).$$

Since  $f$  is homogeneous, we can suppose without loss of generality that if  $x$  exists then  $x_1 \neq \varepsilon$ .

The eigenvalue problem becomes:

$$\begin{cases} \lambda & = f_1(x/x_1), \\ x_2/x_1 & = (f_2/f_1)(x/x_1), \\ \dots & = \dots \\ x_n/x_1 & = (f_n/f_1)(x/x_1), \end{cases}$$

where the division is in minplus sense, that is the subtraction. Denoting  $y = (x_2/x_1, \dots, x_n/x_1)$  and  $g_{i-1}(y) = (f_i/f_1)(0, y)$ , the problem is reduced to the computation of the fixed point problem  $y = g(y)$ . This fixed point gives the normalized eigenvector from which the eigenvalue is deduced by:  $\lambda = f_1(0, y)$ . We note that  $g$  is a non-homogeneous minplus function of  $y$ .

The fixed point problem does not always have a solution, but nevertheless, there are cases where we are able to find one:

- *f is affine in standard algebra.* In this case  $f(x) = Ax + b$ . The homogeneity<sup>3</sup> implies that the kernel of  $A - I_d$  is not empty. When this kernel has one dimension, the eigenvalue is equal to  $\lambda = pb$ , where  $p$  is the normalized ( $p\bar{1} = 1$ ) left standard eigenvector of  $A$  associated to the (standard) eigenvalue 1. But even in this case, all the standard eigenvalues do not have a module necessarily smaller than one, and the dynamical system may be unstable. We note that when all the entries of the matrix  $A$  are nonnegative,  $f$  is monotone nondecreasing, but when there are positive entries and negative entries, the system is not monotone.
- *f is minplus linear:*  $f(x) = A \otimes x$ . In this case the system is monotone.
- *f corresponds to stochastic control.* In this case  $f(x) = D \otimes (Hx)$  where  $H$  is a standard matrix with rows that define discrete probability laws. Such a matrix is called a stochastic matrix. Then the dynamics  $x^{k+1} = f(x^k)$  has the interpretation of a dynamic programming equation associated to a stochastic control problem, and the eigenvalue is the optimal average cost of the corresponding stochastic control problem. Note that in this case,  $x^k$  are components of a the dynamics which can be written

$$y^{k+1} = A \boxtimes y^k, \text{ with } A = \begin{bmatrix} 0 & H \\ D & \varepsilon \end{bmatrix}.$$

- *f corresponds to stochastic games.* In this case  $f = D_1 \odot (D_2 \otimes (Hx))$  where  $\odot$  denotes the maxplus product (obtained by replacing min by max in the minplus matrix product  $\otimes$ ), and  $H$  is a stochastic matrix. This case corresponds to dynamic programming equations associated to stochastic games. In this case,  $f$  is monotone.
- *f has a particular triangular structure* for example:

$$\begin{cases} x^{k+1} = A \otimes x^k, \\ y^{k+1} = B(x^k) \otimes y^k, \end{cases}$$

with  $B(x)$  additively homogeneous of degree 0 but not necessarily monotone. For such systems, it is easy to find the eigenvalue and eigenvector by applying the minplus algebra results. In this case,  $f$  is not always monotone. See [21], [23] for discussions and generalizations.

<sup>3</sup> $A\bar{1} = \bar{1}$  with  $\bar{1}$  the vector with all its entries equal to 1.

In general, it is possible to compute the fixed point using the Newton's method<sup>4</sup>. This method corresponds to the policy iteration when the dynamic programming interpretation holds true. For stochastic control problems, the policy iteration is globally stable. In the game case, the policy iteration is only locally stable.

We may have unstable fixed points which are not accessible by integrating the dynamics. In this case, the eigenvalue, computable by the Newton's method, gives no information on the time asymptotic of the system. When all the fixed points are unstable, we may have a linear growth of the state trajectories. This point is illustrated by the chaotic tent dynamics example given in the next section.

#### F. Growth rate of homogeneous systems

We define the *growth rate*,  $\chi(f)$ , of a dynamic system  $x^{k+1} = f(x^k)$ , where  $x \in \overline{\mathbb{R}}_{\min}^n$ , by the common limit  $\lim_k x_i^k/k$  of all the components  $i$  when this limit exists. In [32] it has been proved, with a special definition of connexity (satisfied for a system defined by  $f(x) = A \boxtimes x$  as long as the graph  $\mathcal{G}(A)$  is strongly connected<sup>5</sup>), that the growth rate and the eigenvalue of an homogeneous and monotone system exist and are equal. Let us show that chaos may appear and that the eigenvalue and growth rate which exist are not equal on a system which is only homogeneous.

Let us consider the homogeneous dynamic system where  $k$  is the time index:

$$\begin{cases} x_1^{k+1} = x_2^k, \\ x_2^{k+1} = (x_2^k)^3 / (x_1^k)^2 \oplus 2(x_1^k)^2 / x_2^k. \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} \lambda x_1 = x_2, \\ \lambda x_2 = x_2^3 / x_1^2 \oplus 2x_1^2 / x_2, \end{cases}$$

where the minplus power exponent must not be confused with a time index.

The solution is  $\lambda = y$  with  $y = x_2/x_1$  satisfying the equation

$$y = y^2 \oplus 2/y^2,$$

<sup>4</sup>we have to solve a piecewise linear system of equations

<sup>5</sup>Here there is an edge from  $i$  towards  $j$  in  $\mathcal{G}(A)$  if  $A_{ji} \neq \varepsilon$  if it is a minplus edge or  $A_{ji} \neq 0$  if it is a standard one.

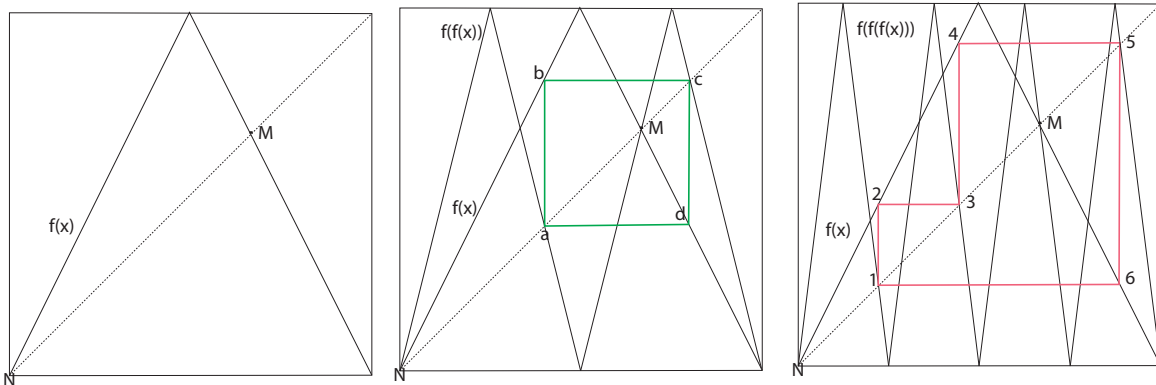


Fig. 2. Cycles of the tent transformation. The abscissa  $x_M$  of  $M$  is a fixed point of  $f$ :  $x_M = f(x_M)$ . The pair  $(x_a, x_c)$  is a cycle of  $f$  composed of two fixed points of  $f \circ f$ :  $x_c = f(x_a)$ ,  $x_a = f(x_c) = f(f(x_a))$ . The triplet  $(x_1, x_3, x_5)$  is a circuit of  $f$  is composed of three fixed points of  $f \circ f \circ f$ :  $x_3 = f(x_1)$ ,  $x_5 = f(x_3) = f(f(x_1))$ ,  $x_1 = f(x_5) = f(f(f(x_1)))$ .

which has the solutions  $y = 0$  and  $y = \frac{2}{3} = 0.66\dots$ . These two solutions are unstable fixed points of the transformation  $f(y) = y^2 \oplus 2/y^2$ . However, the system  $y^{k+1} = f(y^k)$  is chaotic since  $f$  is the tent transform (see [6] for a clear discussion of this dynamics). In Figure-2, we show the graph of  $x \mapsto f(x)$ ,  $x \mapsto f(f(x))$ ,  $x \mapsto f(f(f(x)))$ , their fixed points, and periodic trajectories.

In Figure-3, we show a trajectory for an initial condition chosen randomly with the uniform law on the set  $\{(i-1)/10^5, i = 1, \dots, 10^5\}$ . The diagonal line in the picture is a decreasing order applied to the set  $\{y^k, k = 1, \dots, 10^5\}$ . This line shows that the invariant empirical density is approximately uniform. In fact, with these initial conditions, the trajectories are periodic with a possible long period. It has been proved that the tent iteration has a unique invariant measure absolutely continuous with respect to the Lebesgue measure (the uniform law on  $[0, 1]$ ). Therefore, the system is ergodic. The growth rate

$$(x_1^N - x_1^0)/N = \frac{1}{N} \sum_{k=1}^N (x_1^k - x_1^{k-1}),$$

can be computed by averaging, with respect to the uniform law, an increase in one step:  $f_1(x) - x_1$  with the standard notations (that is  $f_1(x)/x_1 = y$  with the minplus notations). Therefore  $\chi(f) = \int_0^1 y dy = 0.5$ , for almost all initial conditions, which is different from the eigenvalues (0 and  $\frac{2}{3}$ ).

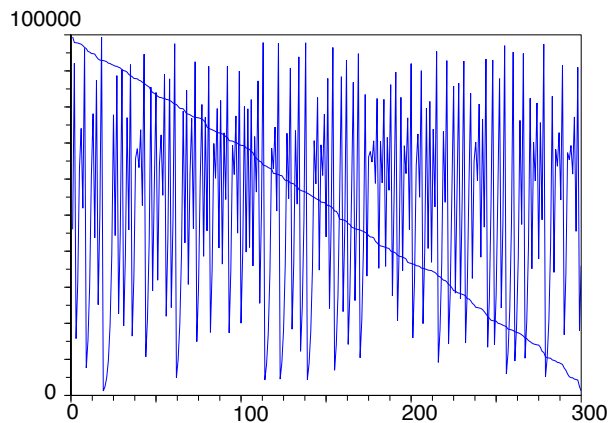


Fig. 3. A tent iteration trajectory  $(10^5 y^k)_{k=1,..,300}$  and its arrangement in decreasing order (almost diagonal line) .

More generally, for a homogeneous system, we can write the system dynamics:

$$x_1^{k+1}/x_1^k = f_1(x^k)/x_1^k = h(y^k), \quad y^{k+1} = g(y^k),$$

with  $y_{i-1}^k = x_i^k/x_1^k$  and  $g_{i-1} = f_i/f_1$  for  $i = 2, \dots, n$ . As long as the  $y^k$  belong to a bounded closed (compact) set for all  $k$ , we remark (after Kryloff and Bogoliuboff) that the set of measures:

$$\left\{ P_{y^0}^N = \frac{1}{N} (\delta_{y^0} + \delta_{g(y^0)} + \dots + \delta_{g^{N-1}(y^0)}) , n \in \mathbb{N} \right\} ,$$

(where  $\delta_a$  denotes the Dirac mass on  $a$ ) is tight. Therefore, we can extract convergent subsequences which converge toward invariant measures  $Q_{y^0}$  that we will call Kryloff-Bogoliuboff invariant measure. Then we can apply the ergodic theorem at the sequence  $(y^k)_{k \in \mathbb{N}}$ . Application of this theorem shows that, for almost all new initial conditions chosen randomly according to the  $Q_{y^0}$ , we have :

$$\chi(f) = \lim_N \frac{1}{N} (x_1^N - x_1^0) = \lim_N \frac{1}{N} \left( \sum_{k=0}^{N-1} h(y^k) \right) = \int h(y) dQ_{y^0}(y) . \quad (2)$$

It may happen that the initial condition  $y^0$  is transient; therefore, it is in the attractive basin of  $Q_{y^0}$  not in the support of  $Q_{y^0}$ . It would be very useful to prove that  $y^0$  is generic (in the sense of Furstenberg[31]), that is the limit exists for  $y^0$ . A priori homogeneous systems do not have the uniform continuity property required to prove the convergence of the Cesaro means for all  $y^0$ . The following classic example show the case of an ergodic system where with non

generic points  $f : x \in \mathbb{T}^1 \rightarrow 2x \in \mathbb{T}^1$  with:  $x^0 = 0.1001111000000000 \dots$  where  $\mathbb{T}^1$  the torus of dimension 1. In Figure-4 we see that the Cesaro does not converge.

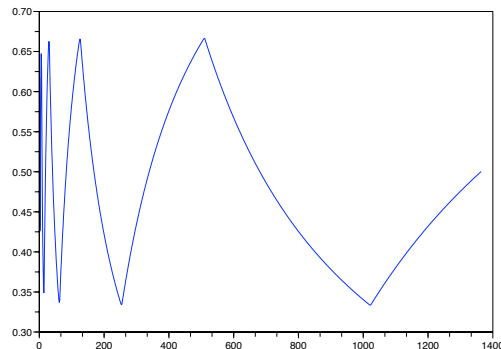


Fig. 4. Plot of  $S(n)$  with:  $S(n) \triangleq \frac{1}{n} \sum_{k=0}^{n-1} x^k$ .

In the case where the compact set is finite, we can apply the ergodicity results on Markov chains with a finite state number to show the convergence of  $P_{y^0}^N$  towards  $Q_{y^0}$ . Instead of the subsequence convergence, this convergence proves the genericity of  $y^0$ .

This discussion is summarized by the following result.

*Theorem 2:* For any additively 1-homogeneous dynamical systems  $x^{k+1} = f(x^k)$ ,  $x^0$  such that  $x_j^k/x_1^k$  stays bounded for all  $j$  and  $k$ , it exists a measure on the initial condition such that the growth rate exists for almost all the initial conditions.

We can also see [2] for construction of invariant measures of stochastic recursions.

Coming back to the tent example, according to the initial value  $y^0$ , the tent iterations  $y^k$  stay in circuits or follow trajectories without circuit (possibly dense in  $[0, 1]$ ). For example, assuming that the initial condition is such that  $y = \frac{2}{5}$ , the trajectory is periodic of period 2. The invariant measure is  $Q_{y^0} = \frac{1}{2}(\delta_{\frac{2}{5}} + \delta_{\frac{6}{5}})$ . The growth rate is  $\frac{4}{5}$ , which is again different from the eigenvalues 0 and  $\frac{2}{3}$ . Moreover, it can be shown that for all initial conditions with a finite binary development (this set contains all the float numbers of computers), the trajectory stays in the unstable fixed point 0 after a finite number of steps. That is, for a dense set of initial conditions the invariant measure is  $\delta_0$  and the growth rate is 0.



### III. PETRI NET DYNAMICS

#### A. Autonomous Petri nets

Let us give, in min-plus-times algebra, a presentation of timed continuous Petri nets with weights. The weight can be negative and the numbers of tokens are not necessary integer (in continuous Petri nets, what we call tokens are in fact fluid amounts).

A Petri net  $\mathcal{N}$  is a graph with two sets of nodes (the *transitions*  $\mathcal{Q}$  (with  $|\mathcal{Q}|$  elements) and the *places*  $\mathcal{P}$  (with  $|\mathcal{P}|$  elements)) and two sorts of edges, (the *synchronization edges* (from a place to a transition) and the *production edges* (from a transition to a place)).

A minplus  $|\mathcal{Q}| \times |\mathcal{P}|$  matrix  $D$ , called *synchronization*<sup>6</sup> *matrix* is associated to the synchronization edges.  $D_{qp} = a_p$  if there exists an edge from the place  $p \in \mathcal{P}$  to the transition  $q \in \mathcal{Q}$ , and  $D_{qp} = \varepsilon$  elsewhere, where  $a_p$  is the *initial marking* of the place  $p$ , which is, graphically, the number of tokens in  $p$ . We suppose here that the sojourn time in all the places is one unit of time<sup>7</sup>.

A standard algebra  $|\mathcal{P}| \times |\mathcal{Q}|$  matrix  $H$ , called *production*<sup>8</sup> *matrix* is associated with the production edges. It is defined by  $H_{pq} = m_{pq}$  if there exists an edge from  $q$  to  $p$ , and 0 elsewhere, where  $m_{pq}$  is the *multiplicity* of the edge<sup>9</sup>.

Therefore, a Petri net is characterized by the quadruple:

$$(\mathcal{P}, \mathcal{Q}, H, D) .$$

A Petri net is a dynamic system in which the token (fluid) evolution is partially defined by the transition firings, saying that a transition can fire as soon as all its upstream places contain a positive quantity of tokens (fluid) having stayed at least one unit of time. When a transition fires, it consumes a quantity of tokens (fluid) equal to the minimum of all the available quantities being in the upstream places. Cumulating the firings done up to present time defines the *cumulated*

<sup>6</sup>Decision matrix in stochastic control.

<sup>7</sup>When different integer sojourn times are considered, an equivalent Petri net with a unique sojourn time can be obtained by adding places and transitions and solving the implicit relations.

<sup>8</sup>Hazard matrix in stochastic control

<sup>9</sup>Here the multiplicity appears only with the output transition edges. The multiplicity of input transition edges is supposed to be always equal to one. Looking at the more general case [10], we see that we do not lose generality by doing so (the dynamics class obtained is the same)

*transition firing* of the transition. The firing produces a quantity of tokens (fluid) in each downstream place equal to the firing of the transition multiplied by the multiplicity of the corresponding production edge. If the multiplicity of a production edge, going from  $q$  to  $p$ , is negative, the firing of  $q$  consumes tokens (fluid) of  $p$ .

A general Petri Net defines constraints on the transition firing. Denoting by  $q^k$  the cumulated firings of transitions  $q \in \mathcal{Q}$  up to instant  $k$ , they satisfy the constraints:

$$\min_{p \in q^{in}} \left[ a_p + \sum_{q \in p^{in}} m_{pq} q^{k-1} - \sum_{q \in p^{out}} q^k \right] = 0, \quad \forall q \in \mathcal{Q}, \quad \forall k, \quad (3)$$

where  $p \in \mathcal{P}$  is a place of the Petri Net;  $q^{in} \subset \mathcal{P}$  [resp.  $q^{out} \subset \mathcal{P}$ ] denotes set of places upwards [resp. downwards] the transition  $q$ ; and  $p^{in} \subset \mathcal{Q}$  [resp.  $p^{out} \subset \mathcal{Q}$ ] denotes the set of transitions upwards [resp. downwards] the place  $p$ .

Indeed, being at time  $k$ , we know (from the firing definition of transitions) that after the firing (which is instantaneous), there is at least one place upstream of any transition in which no token entered before time  $k - 1$ . For each transition  $q$ , the equation (3) computes the number of tokens which have stayed at least one unit of time in each place  $p \in q^{in}$ . The equation expresses that at least one place is empty and the others have nonnegative numbers of tokens.

As long as there is more than one edge leaving a place, the trajectory of the system is not well-defined because we do not know the path of a token leaving this place.

In the case of a *deterministic* Petri net (generally called conflict free Petri net) where all the places have only one downstream edge, the dynamics are well-defined, meaning there is no token consumption conflicts between the transitions downstream of each place<sup>10</sup>. Then, denoting by  $Q = (q^k)_{q \in \mathcal{Q}, k \in \mathbb{N}}$  the vector of sequences of cumulated firing quantities of transitions, and by  $P = (p^k)_{p \in \mathcal{P}, k \in \mathbb{N}}$  the vector of sequences of cumulated token quantities arrived in the places at time  $k$ , we have:

$$\begin{bmatrix} P^{k+1} \\ Q^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & H \\ D & \varepsilon \end{bmatrix} \boxtimes \begin{bmatrix} P^{k+1} \\ Q^k \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} HQ^k \\ D \otimes P^{k+1} \end{bmatrix}. \quad (4)$$

In Equation-4,  $P^{k+1}$  counts the number of tokens available (for firing in the downstream transitions) at time  $k + 1$  coming from the upstream transition firings. It is obtained by summing

<sup>10</sup>In the non-deterministic case, we have to specify the rules which resolve the conflicts by, for example, giving priorities to the consuming transitions or by imposing ratios to be followed. As long as these rules are added, the initial non-deterministic Petri net becomes a deterministic one.

the weighted firing numbers of the transitions upstream the places since the tokens are supposed to stay at least one unit of time in the places.

The part about  $Q^{k+1}$  in Equation-4 says that the transition firing numbers at time  $k + 1$  are equal to the minimal token numbers in the places upstream the transitions. To obtain these quantities we have only to add the token numbers in the place at initial time to the numbers entered by the firings that are the entries of  $P^{k+1}$ .

From these dynamics, we deduce the dynamics of the cumulated firing quantities by eliminating the place variables. We deduce the dynamics of the cumulated token quantities by eliminating the transition variables.

$$Q^{k+1} = D \otimes (HQ^k), \quad P^{k+1} = H(D \otimes P^k) .$$

In the case of *event graphs* (particular deterministic Petri nets where all the multiplicities  $m_{pq}$  are equal to 1 and all the places have exactly one edge upstream), the dynamics are linear in the minplus sense given by:

$$Q^{k+1} = A \otimes Q^k ,$$

where  $A_{q'q} = a_p$  with  $p$  the unique place between  $q$  and  $q'$ .

### B. Deterministic Petri nets

Using the negative weights and/or fixing a routing policy it is possible to transform a Petri net with conflicts in a deterministic Petri net. Let us discuss these points more precisely on the simple system given in the first picture of Figure-5.

The incomplete dynamics of this system can be written in minplus algebra<sup>11</sup>:

$$q_4^k q_3^k = a q_1^{k-1} q_2^{k-1} . \tag{5}$$

Clearly  $q_3$  and  $q_4$  are not defined uniquely. In the following two examples, we complete the dynamics in two different ways. These two ways are equally useful for traffic applications as we will see later.

<sup>11</sup>Which means in standard algebra:  $q_4^k + q_3^k = a + q_1^{k-1} + q_2^{k-1}$ .

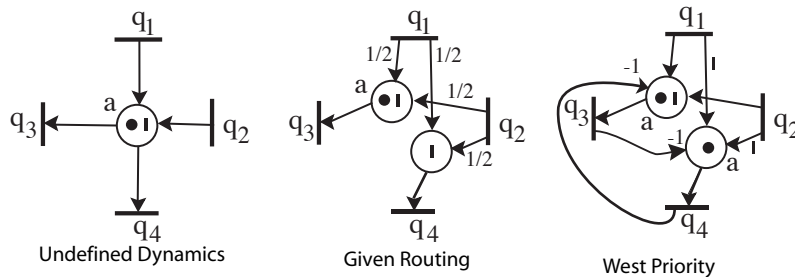


Fig. 5. The nondeterministic Petri net, given in the left figure, is made clear by: – choosing a routing policy:  $1/2$  (to read in standard algebra) towards  $q_3$ ,  $1/2$  towards  $q_4$  in the central figure, – giving top priority to  $q_3$  against  $q_4$  in the right figure (where the time shift given by the sticks are not anymore only in the places but also on the edges).

- By specifying the routing policy (for example we choose arbitrarily that half of the total tokens available are given to  $q_3$  and half to  $q_4$ , see [10] for results on the general routing case)<sup>12</sup>:

$$q_4^k = \sqrt{q_1^{k-1} q_2^{k-1}}, \quad q_3^k = a q_4^k .$$

The minplus product of the two equations gives the constraint (5).

- By choosing a priority rule (top priority to  $q_3$  against  $q_4$ )<sup>13</sup>:

$$q_3^k = a q_1^{k-1} q_2^{k-1} / q_4^{k-1}, \quad q_4^k = a q_1^{k-1} q_2^{k-1} / q_3^k .$$

The last equation implies that the initial constraint (5) is satisfied. We see that the negative weights on  $q_4^{k-1}$  and on  $q_3^k$  are essential to express this priority.

In the two cases, we obtain a *degree one homogeneous minplus* system.

### C. Input-Output Petri nets

We can define a Petri net with inputs and outputs in the following way. We partition the transition set in three parts  $(\mathcal{V}, \mathcal{Q}, \mathcal{Z})$ : the input set  $\mathcal{V}$ , the state set  $\mathcal{Q}$  and the output set  $\mathcal{Z}$ . We do the same thing for the place set and we get the three parts  $(\mathcal{U}, \mathcal{P}, \mathcal{Y})$ . The inputs are

<sup>12</sup>Which means in standard algebra:  $q_4^k = \frac{1}{2} (q_1^{k-1} + q_2^{k-1})$ ,  $q_3^k = a + q_4^k$ .

<sup>13</sup>Which means in standard algebra:  $q_3^k = a + q_1^{k-1} + q_2^{k-1} - q_4^{k-1}$ ,  $q_4^k = a + q_1^{k-1} + q_2^{k-1} - q_3^k$ .

the transitions [resp. places] without upstream edges. The outputs are the ones without output edges. Then the dynamics can be rewritten:

$$\begin{bmatrix} P^{k+1} \\ Q^{k+1} \\ Y^{k+1} \\ Z^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & A & 0 & B \\ C & \varepsilon & D & \varepsilon \\ 0 & E & 0 & 0 \\ F & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \boxtimes \begin{bmatrix} P^{k+1} \\ Q^k \\ U^{k+1} \\ V^k \end{bmatrix} \triangleq \begin{bmatrix} AQ^k + BV^k \\ C \otimes P^{k+1} \oplus D \otimes U^{k+1} \\ EQ^k \\ F \otimes P^{k+1} \end{bmatrix}. \quad (6)$$

These dynamics, denoted by  $S$  and defined by the matrices  $(A, B, C, D, E, F)$ , associate the output signals  $(Y^k, Z^k)_{k \in \mathbb{N}}$  to the input signals  $(U^k, V^k)_{k \in \mathbb{N}}$ . We write  $(Y, Z) = S(U, V)$ . These systems can be considered as a special case of an extension of the generalized system described in the minplus section. In this special case, some blocks are null and an implicit part can appear.

The three operations (parallel, series, and feedback compositions) described previously can be extended easily to this case.

## IV. TRAFFIC APPLICATION

### A. Traffic on a circular road

Let us recall the simplest model to derive the fundamental traffic diagram on a single road. The best way to obtain this diagram is to study the stationary regime on a circular road with a given number of vehicles<sup>14</sup>. We present two ways to obtain this diagram: one is by logical deduction from an exclusion process point of view (it shows clearly the presence of two distinct phases), and the other by computing the eigenvalue of a minplus system derived from a simple Petri net modeling of the road (this way will be extended to the case of roads with junctions). In the following, the road is cut in  $m$  sections which can contain at most one vehicle.

### B. Exclusion process modeling

Following [7], we can consider the dynamical system defined by the rule  $10 \rightarrow 01$  applied to a binary word  $w$ . The word  $w_k$  describes the vehicle positions at instant  $k$  on a road cut

<sup>14</sup>We consider that the stationary regime of the circular road is reached locally on a standard road when its density is constant at the considered zone.

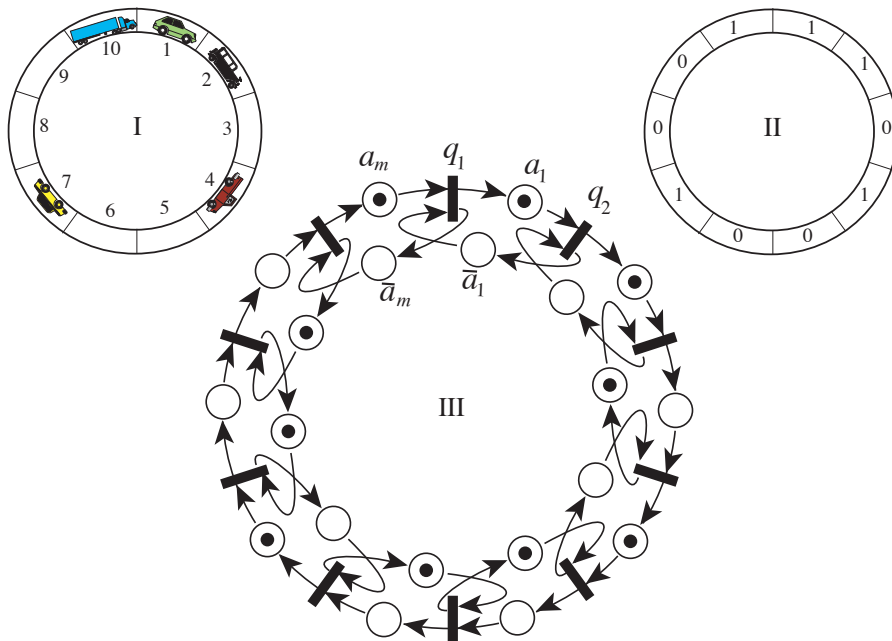


Fig. 6. On the top-left side, a circular road cut in sections. On the top-right side, its exclusion process, where 1 means that the section is occupied. On the middle, its Petri net representation where the ticks (1 time delay) in each place are not represented.

in sections (each bit representing a section, 1 meaning occupied and 0 meaning free, see II in Figure-6). Let us take an example:

$$w_1 = 1101001001, \quad w_2 = 1010100101,$$

$$w_3 = 0101010011, \quad w_4 = 1010101010,$$

$$w_5 = 0101010101.$$

Let us define: – the *density*  $\rho$  by the number of vehicles  $n$  divided by the number of sections  $m$ :  $\rho = n/m$ , – the *flow*  $\varphi(t)$  at time  $t$  by the number of vehicles going one step forward at time  $t$  divided by the number of sections. Then the *fundamental traffic diagram* gives the relation between  $\varphi(t)$  and  $\rho$ .

If  $\rho \leq 1/2$ , after a transient period, all the vehicle groups split off, and then all the vehicles can move forward without other vehicles in the way, and we have:

$$\varphi(t) = \varphi = n/m = \rho .$$

If  $\rho \geq 1/2$ , the free place groups split off after a finite time and move backward without other free places in the way. Then  $m - n$  vehicles move forward and we have:

$$\varphi(t) = \varphi = (m - n)/m = 1 - \rho .$$

*Theorem 3:*

$$\exists T : \forall t \geq T \quad \varphi(t) = \varphi = \begin{cases} \rho & \text{if } \rho \leq 1/2 , \\ 1 - \rho & \text{if } \rho \geq 1/2 . \end{cases}$$

*Proof:* This result has been proved in [7]. Let us give the idea of the proof. We only have to prove that after a finite number of steps, the separations of vehicles or holes appear.

Assuming that the density is not greater than  $1/2$ , let us look what happens at a cluster of at least two vehicles denoted by A. There are two cases:

- The cluster behind A is separated from A by one hole, we have the configuration 10A0. Then at the next step we have 0A01. The cluster A has gone backward by one place.
- The cluster behind is separated from A by more than one hole, we have the configuration 00A0. Then at the next step, the size of A has been reduced by one.

Then after a finite number of steps, bounded by the number of places, the size of each cluster of vehicles is reduced strictly. Indeed, the density being not greater than  $1/2$ , and individual vehicles going forward, there is cluster of vehicles that must meet a cluster of at least two holes.

For a density larger than  $1/2$ , we follow the cluster of holes instead of vehicles. We can show by the same arguments that their sizes decrease or they go forward. Therefore after a finite number of steps the holes are separated by vehicles. ■

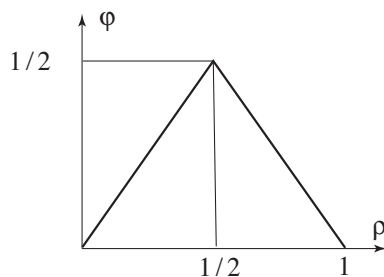


Fig. 7. The fundamental traffic diagram showing the dependence on the average flow with respect of the vehicle density.

### C. Event Graph modeling

Consider the Petri net given in III of Figure-6 which describes the same dynamics in a different way. In fact, this Petri net is an event graph. Therefore, the dynamics are linear in minplus algebra. The vehicle number entered in the section  $s$  before time  $k$  is denoted  $q_s^k$ . The initial vehicle position is given by the booleans  $a_s$  which take the value 1 when the cell contains a vehicle and 0 otherwise.

We use the notation  $\bar{a} = 1 - a$ . Then, the dynamics are given by:

$$q_s^{k+1} = \min\{a_{s-1} + q_{s-1}^k, \bar{a}_s + q_{s+1}^k\},$$

which can be written linearly in minplus algebra:

$$q_s^{k+1} = a_{s-1}q_{s-1}^k \oplus \bar{a}_s q_{s+1}^k,$$

where the index addition is done modulo  $m$ .

*Theorem 4:* The average transition speed (car flow)  $\varphi$  depends on the car density  $\rho$  according to the law:

$$\varphi = \min(\rho, 1 - \rho) \quad \square$$

*Proof:* This event graph has three kinds of elementary circuits: the outside circuit with average weight  $n/m$ ; the inside circuit with average weight  $(m - n)/m$ ; and the circuits on which some steps are made forward and then back with the average mean  $1/2$ . Therefore, using Theorem 1, its eigenvalue is

$$\varphi = \min(n/m, (m - n)/m, 1/2) = \min(\rho, 1 - \rho),$$

which gives the average speed as a function of the vehicle density since the minplus eigenvalue is equal to  $\lim_t x_i(t)/t = \varphi$  for all  $i$ .

### D. Traffic on two roads with one junction

Before discussing the dynamics of the town, let us study in detail the case of two circular roads with a junction (see the top-right side of Figure-8).

A first trial is to consider the Petri net given in the middle of Figure-8. This Petri net is not an event graph. It is a general non deterministic Petri Net.

We can write the dynamics of this Petri net using Equation-3, but these equations do not uniquely determine the trajectories of the system. We have two places with two outgoing edges,



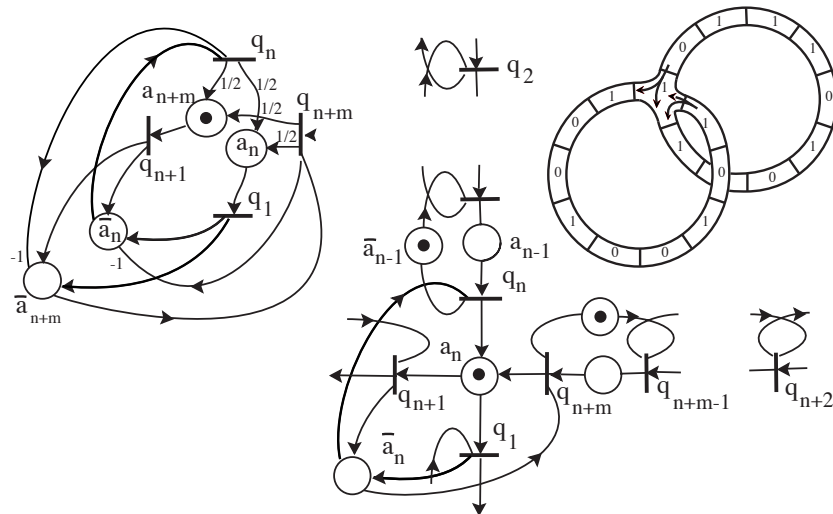


Fig. 8. A junction with two circular roads cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left). The ticks representing the time delays present in each place are not represented.

$a_n$  and  $\bar{a}_n$ . At place  $a_n$ , we have to specify the *routing policy* giving the proportion of cars going towards  $y_2$  and the proportion going towards  $y_3$ . At place  $\bar{a}_n$ , we may follow *the first arrived the first served rule with the right priority* when two vehicles want to enter in the junction simultaneously. Using Petri net with negative weights, we obtain the Petri Net of Figure-8 where the junction is described precisely in the top-left part of the figure. In the place  $a_n$  [resp.  $a_{n+m}$ ] are accumulated the cars going towards West [resp. towards South]. In the place  $\bar{a}_n$  [resp.  $\bar{a}_{n+m}$ ] are accumulated the authorizations to enter in the junction from North [resp. from East]. These authorizations are obtained by subtracting the authorizations to enter from East [resp. from North] from the total number of authorizations to enter in the junction. See Section III-B to have a precise description of the management of the right priority. We remark that we have obtained a deterministic (conflict free) Petri Net with some negative weights but not an event graph with negative tokens.

The dynamics describing the evolution of the vehicle number entered in section  $s$  before time  $k$  denoted  $q_s^k$  can be obtained immediately from the Petri net in top left corner of Figure-8 using the minplus notation discussed in Section-II-B.

$$\begin{cases} q_i^{k+1} = a_{i-1}q_{i-1}^k \oplus \bar{a}_i q_{i+1}^k, & i \neq 1, n, n+1, n+m, \\ q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k \oplus a_{n-1} q_{n-1}^k, \\ q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1} \oplus a_{n+m-1} q_{n+m-1}^k, \\ q_1^{k+1} = a_n \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_1 q_2^k, \\ q_{n+1}^{k+1} = a_{n+m} \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_{n+1} q_{n+2}^k, \end{cases} \quad (7)$$

where the entries satisfy the following constraints (written with the standard notations):

- $0 \leq a_i \leq 1$  for  $i = 1, \dots, n+m$ . These initial markings give the presence, 1, or absence, 0, of a vehicle in the road sections. However, here we see vehicles as fluid and can relax this integer constraint. Moreover, the transition firing cuts the tokens, and if we want to see the systems after some firings, there is not necessarily an integer number of tokens in a place. Therefore, it is better to accept real numbers of tokens belonging to the  $[0, 1]$  interval;
- $\bar{a}_i = 1 - a_i$  for  $i \neq n, n+m$  they give the initial free spaces in the places;
- $a_n + a_{n+m} \leq 1$  the maximum number of cars in the junction is 1;
- $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$  give the free place in the junction.

We remark that the system is homogeneous of degree 1 and that it is easy to write the dynamics using the generalized matrix product. Indeed, all the “monomials” appearing in the right hand side of System 7 (for example  $\sqrt{q_n^k q_{n+m}^k}$ ) are linear in the standard algebra and can be computed by a standard matrix product from the  $q$  vector. Then it is easy to obtain the complete right hand side from all the appearing “monomials” by a minplus matrix product. This matrix form is very useful to simulate the system in ScicosLab [46] since the minplus matrix product is implemented in this software. We remark also that the “monomials” like  $q_1^k q_{n+1}^k / q_{n+m}^k$  introduce negative entries in the standard matrices. Therefore the dynamics is not monotonic.

We can prove the existence of a growth rate thanks to the following two results.

*Theorem 5:* The trajectories of the states of the junction dynamics (7),  $(q_i^k)_{k \in \mathbb{N}}$ , starting from 0, are nonnegative and nondecreasing for all  $i$   $\square$

*Proof:* Computing  $q^1$  using the fact that all the  $a_i$  and  $\bar{a}_i$  are nonnegative, it is clear that  $q^1 \geq 0$ . Let us prove by induction that the trajectories are nondecreasing. It is true for  $k = 1$ . We suppose that it is true for  $k$ , that is  $q^k \geq q^{k-1}$ , and we prove that it is also true for  $k + 1$ , that is  $q^{k+1} \geq q^k$ . Rewrite (7)  $q_i^{k+1} = f_i(q^k)$  for  $i = 1, \dots, n+m$ . The functions  $f_i$  for  $i \neq n, n+m$

are nondecreasing. Therefore, for such an  $i$ , we have:

$$q_i^{k+1} = f_i(q^k) \geq f_i(q^{k-1}) \geq q_i^k,$$

using first the induction hypothesis and then the dynamics definition.

Let us prove the nondecreasing property of  $q_n$ .

- If  $q_n^{k+1} = a_{n-1}q_{n-1}^k$  we have

$$q_n^{k+1} = a_{n-1}q_{n-1}^k \geq a_{n-1}q_{n-1}^{k-1} \geq f_n(q^{k-1}) = q_n^k.$$

- If  $q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$ , using the dynamics, we have  $q_{n+m}^k \leq \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1} / q_n^k$ . Therefore,

$$q_n^{k+1} \geq \bar{a}_n q_1^k q_{n+1}^k / \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1}$$

which gives  $q_n^{k+1} \geq q_n^k$  using the induction hypothesis and the assumption  $\bar{a}_n = \bar{a}_{n+m}$ .

The nondecreasing property of  $q_{n+m}$  is proved in the same way. If  $q_{n+m}^{k+1} = a_{n+m-1}q_{n+m-1}^k$ , we have  $q_{n+m}^{k+1} = a_{n+m-1}q_{n+m-1}^k \geq a_{n+m-1}q_{n+m-1}^{k-1} \geq f_{n+m}(q^{k-1}) = q_{n+m}^k$ . If  $q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1}$ , we have  $q_n^{k+1} \leq \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$  using the dynamics. Therefore,  $q_{n+m}^{k+1} \geq \bar{a}_{n+m} q_{n+m}^k / \bar{a}_n$ , which gives the result using  $\bar{a}_n = \bar{a}_{n+m}$ . ■

*Theorem 6:* The distances between any pair of states stay bounded.

$$\exists c_1 : \sup_k |q_i^k - q_j^k| \leq c_1, \forall i, j.$$

Moreover

$$\forall T, \exists c_2 : \sup_k |q_i^{k+T} - q_i^k| \leq c_2 T, \forall i \quad \square$$

*Proof:* This result comes from the following inequalities (written in minplus algebra) deduced from the dynamics and the nondecreasing property of the trajectories:

$$q_i^{k+1} \leq a_{i-1}q_{i-1}^k, i \neq 1, n+1$$

$$q_1^{k+1} \leq a_n \sqrt{q_n^k q_{n+m}^k} \leq a_n \sqrt{q_n^{k+n} q_{n+m}^k} \leq a_n \sqrt{b_1^{n-1} q_1^{k+1} q_{n+m}^k} \Rightarrow q_1^{k+1} \leq a_n b_1^n q_{n+m}^k,$$

with  $b_j^k = \bigotimes_{i=j}^k a_i$ .

$$q_{n+1}^{k+1} \leq a_{n+m} \sqrt{q_n^k q_{n+m}^k} \leq a_{n+m} \sqrt{q_n^k q_{n+m}^{k+m}} \leq a_{n+m} \sqrt{q_n^k b_n^{n+m-1} q_{n+1}^{k+1}} \Rightarrow q_{n+1}^{k+1} \leq a_{n+m} b_n^{n+m} q_n^k.$$

Therefore we have:

$$q_n^k \leq a_{n-1}q_{n-1}^{k-1} \leq \dots \leq (b_1^n)^2 q_{n+m}^{k-n} \leq (b_1^n)^2 a_{n+m-1} q_{n+m-1}^{k-n-1} \leq \dots \leq (b_1^{n+m})^2 q_n^{k-n-m}.$$

The result follows from these inequalities which give bounds for all the the distances between two states and between the same state but at different times. ■

Using this theorem and Theorem 2, we show an existence theorem of the growth rate of (7) which has the interpretation of the average traffic flow.

*Theorem 7:* There exists an initial distribution on  $(q_j^0/q_1^0)_{j=2,n+m}$  the Kryloff-Bogoljuboff invariant measure such that the average flow

$$\chi = \lim_k q_i^k/k, \quad \forall i,$$

of the dynamical system (7) exists almost everywhere □

This result is not completely satisfactory. We would like to have the existence of the growth rate for the initial condition  $q_i^0 = 0$  for all  $i$ . Nevertheless, we note that the numerical approximation obtained by simulation of the growth rate always exists. Indeed, in this case, the number of the approximated states obtained by floating number approximation is finite since they belong to a bounded set<sup>15</sup>. The existence of the growth rate can be proved using the Cesaro-convergence of the probability to be in a state toward an invariant measure in the case of finite Markov chains. At this point, it is also useful to recall that the continuous Petri net model used here is an approximation of a more realistic discrete Petri net.

*Theorem 8:* Starting from  $q^0 = 0$ , if an average growth rate  $\chi$  exists, then  $\chi \leq 1/4$ .

*Proof:* From the dynamics (7) we have :

$$\begin{aligned} q_{n+m}^{k+1} &\leq \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1}, \\ q_1^{k+1} &\leq a_n \sqrt{q_n^k q_{n+m}^k}, \\ q_{n+1}^{k+1} &\leq a_{n+m} \sqrt{q_n^k q_{n+m}^k}. \end{aligned}$$

then by summing these inequalities we get :

$$(q_1^{k+1} - q_1^k) + (q_n^{k+1} - q_n^k) + (q_{n+1}^{k+1} - q_{n+1}^k) + (q_{n+m}^{k+1} - q_{n+m}^k) \leq 1.$$

Hence, when  $k \rightarrow +\infty$ , and taking into account Theorem-6, we obtain  $4\chi \leq 1$  ■

<sup>15</sup>We suppose here that the approximation does not destroy the fact that the states stay in a bounded set .

### E. Eigenvalue Existence of the Junction Dynamics

Let us consider the eigenvalue problem associated to the dynamics (7). It is defined as finding  $\lambda$  and  $q$  such that:

$$\left\{ \begin{array}{l} \lambda q_i = a_{i-1} q_{i-1} \oplus \bar{a}_i q_{i+1}, \quad i \neq 1, n, n+1, n+m, \\ \lambda q_n = \bar{a}_n q_1 q_{n+1} / q_{n+m} \oplus a_{n-1} q_{n-1}, \\ \lambda q_{n+m} = \bar{a}_{n+m} q_1 q_{n+1} / (\lambda q_n) \oplus a_{n+m-1} q_{n+m-1}, \\ \lambda q_1 = a_n \sqrt{q_n q_{n+m}} \oplus \bar{a}_1 q_2, \\ \lambda q_{n+1} = a_{n+m} \sqrt{q_n q_{n+m}} \oplus \bar{a}_{n+1} q_{n+2}, \end{array} \right. \quad (8)$$

with:  $0 \leq a_i \leq 1$  for  $i = 1, \dots, n+m$ ,  $\bar{a}_i = 1 - a_i$  for  $i \neq n, n+m$ ,  $a_n + a_{n+m} \leq 1$  and  $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$ .

The eigenvalue problem can be solved explicitly.

*Theorem 9:* The nonnegative eigenvalues  $\lambda$  as a function of the density  $d$ , written in the standard algebra, is given by:

$d$	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta, \gamma) < d < \max(\beta, \gamma)$	$\gamma \leq d \leq 1$
$\lambda$	$(1 - \rho)d$	$1/4$	$(r - (1 - \rho)d) / (2r - 1 + 2\rho)$	$0$

with  $N = n + m$ ,  $\rho = 1/N$ ,  $r = m/N$  and  $d = (\sum_{i=1}^{n+m} a_i) / (N - 1)$  the density of vehicles,  $\alpha = 1/4(1 - \rho)$ ,  $\beta = (r + 1/2 - \rho) / 2(1 - \rho)$  and  $\gamma = r / (1 - \rho)$   $\square$

If  $r > 1/2$  when  $N$  is large, the positive eigenvalue is unique and has the simple approximation:

$$\lambda \simeq \max \left\{ 0, \min \left\{ d, \frac{1}{4}, \frac{r - d}{2r - 1} \right\} \right\}.$$

*Proof:* The whole proof is available in [22] (where the role of  $m$  and  $n$  are inverted). Only a sketch of the proof is given in this section. The proof has two parts. The first part consists of reducing the problem to a generalized eigenvalue problem in a four dimensional space (this part is easy and is not given here, it is available in [22]). The second part consists of a verification of the generalized minplus eigenvalue system of equations since we give explicit formulas for all the eigenelements (this part is given in the Appendix, and is also available in details in [22]). The eigenelements have been obtained by solving explicitly the homogeneous affine systems with five unknowns achieving the minimum in the reduced system. Numerical simulations have suggested the affine system achieving the minimum. Knowing it we had only to verify the inequalities proving that it achieves actually the minimum.

Since we want the eigenvalue as a function of the density, we cannot use a numerical approach. Instead, we have to find explicit formulas which are piecewise affine. This is the main difficulty of the problem.

After verifying that  $\lambda \leq 1/4$ , by elimination of  $q_i$ ,  $i \neq 1, n, n+1, n+m$ , thanks to the minplus linearity of the first equation of (8), by elimination we obtain the closed set of equations defining  $q_i$ ,  $i = 1, n, n+1, n+m$  see Lemma 1 in the annex:

$$\begin{cases} q_n = (\bar{a}_n/\lambda)q_1q_{n+1}/q_{n+m} \oplus (b_n/\lambda^{n-1})q_1, \\ q_{n+m} = (\bar{a}_{n+m}/\lambda^2)q_1q_{n+1}/q_n \oplus (b_m/\lambda^{m-1})q_{n+1}, \\ q_1 = (a_n/\lambda)\sqrt{q_nq_{n+m}} \oplus (\bar{b}_n/\lambda^{n-1})q_n, \\ q_{n+1} = (a_{n+m}/\lambda)\sqrt{q_nq_{n+m}} \oplus (\bar{b}_m/\lambda^{m-1})q_{n+m}, \end{cases} \quad (9)$$

where  $b_n = \bigotimes_{i=1}^{n-1} a_i$  is the number of cars in the street with priority;  $\bar{b}_n = \bigotimes_{i=1}^{n-1} \bar{a}_i$  is the number of free places in the street with priority;  $b_m = \bigotimes_{i=n+1}^{n+m-1} a_i$  is the number of cars in the street without priority; and  $\bar{b}_m = \bigotimes_{i=n+1}^{n+m-1} \bar{a}_i$  is the number of free places in the street without priority.

Observing from numerical simulations that four phases exist, it is possible to specify their domains analytically, and give their physical interpretations by observing the asymptotic regimes:

- *Free moving.* When the density is small,  $0 \leq d \leq \alpha$  after a finite time, all the vehicles move freely.
- *Saturation.* When  $\alpha \leq d \leq \beta$ , the junction is used at its maximal capacity without being bothered by downstream vehicles.
- *Recession.* When  $\beta < d < \gamma$ , the crossing is fully occupied, but vehicles sometimes cannot leave the crossing because the road they want to enter is crowded. When  $\gamma < \beta$ , three eigenvalues exist on the interval  $[\gamma, \beta]$ . In this case, the system is blocked.
- *Freeze.* When  $\gamma \leq d \leq 1$ , the road without priority is full of vehicles. No vehicle can leave this road when the vehicle being in the junction wants to enter.

Using these understanding of the phases we have been able to guess where the minimum are achieved and then to find the solution by solving a standard linear system of 4 equations and 5 unknown for each phase. In the Appendix, we show that Table I (written in standard algebra) gives the eigenvalue and eigenvector formulas as a function of the density  $d$  for each phase.

	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta, \gamma) < d < \max(\beta, \gamma)$	$\gamma \leq d \leq 1$
$\lambda$	$(1 - \rho)d$	$\frac{1}{4}$	$\frac{r - (1 - \rho)d}{2r - 1 + 2\rho}$	0
$q_n$	$b_n - (n - 1)\lambda$	$b_n - (n - 1)\lambda$	$b_n - (n - 1)\lambda$	$\bar{a}_{n+m} + \bar{b}_m$
$q_{n+m}$	$(n + 1)\lambda - 2a_n - b_n$	$(n + 1)\lambda - 2a_n - b_n$	$(n + 1)\lambda - 2a_n - b_n$	$-2a_n - \bar{a}_{n+m} - \bar{b}_m$
$q_1$	0	0	0	0
$q_{n+1}$	$a_{n+m} - a_n$	$a_{n+m} - a_n$	$4\lambda - 1 + a_{n+m} - a_n$	$-2a_n - \bar{a}_{n+m}$

TABLE I

THE EIGENVALUES AND EIGENVECTORS AS FUNCTION OF THE CAR DENSITIES.

In Figure-9, we show the fundamental diagram obtained by simulation (using the maxplus arithmetic of the ScicosLab software [46]) for a particular relative size  $r$  of the two roads and the eigenvalue  $\lambda$  given in Table I. We see clearly the four phases described in the proof of the eigenvalue formula. On this figure, we see also that the growth rate and the eigenvalue are very close to each other at least for three of the four phases.

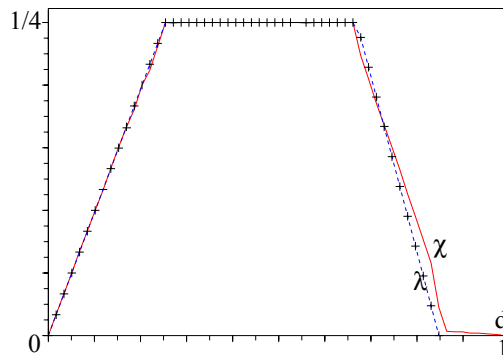


Fig. 9. The traffic fundamental diagram  $\chi(d)$  when  $r = 5/6$  (continuous line) obtained by simulation and its comparison with the eigenvalue  $\lambda(d)$  given in Table I.

### F. Regular City Modeling.

We can generalize the modeling approach used in the case of one junction to derive the fundamental diagram of a regular town on a torus described in Figure-11 (left). We build the

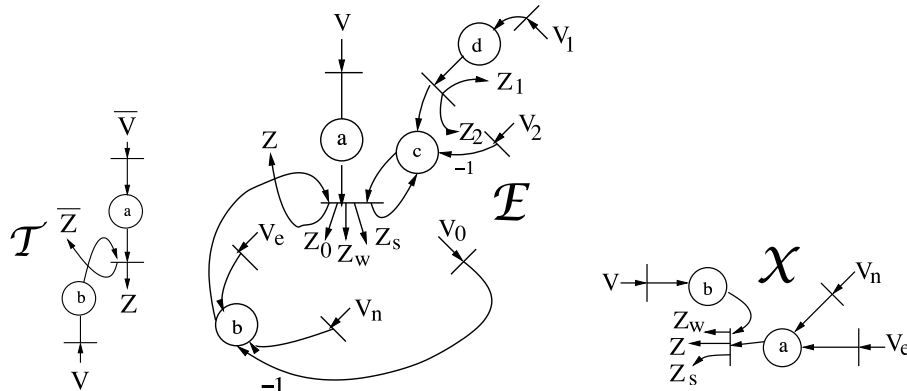


Fig. 10. The three elementary Petri nets with which it is possible to build the Petri net of a regular set of roads on a torus by composition.

model by combining three elementary systems (representing one section (T) , a junction input (E), and a junction output (X)) with the composition operators described in the generalized system theory subsection of the minplus section. The details can be found in [21].

The asymptotic vehicle distribution for a small town composed of two North-South, South-North, East-West and West-East avenues is given in Figure-11 (right). The fundamental diagram presents a four-phase shape analogous to the case of two roads with one junction. In this more general case, the role of the non-priority road is played by a circuit of non-priority roads which blocks the whole system when the circuit is full.

## V. CONCLUSION

The Petri net modeling of traffic in town can be done thanks to the introduction of negative weights on output transition edges. The dynamics have a nice degree one homogeneous minplus property but are no longer monotone. This loss of monotonicity implies that the eigenvalue and the growth rate are no longer equal. Experimental results show that they are close in the case of two roads with one crossing where we are able to solve explicitly the eigenvalue problem and compute numerically the growth rate. The fundamental traffic diagram, which gives the dependence of the average car-flow (given by the growth rate) on the density, presents four phases that have traffic interpretations.

This set of 1-homogeneous minplus systems seems to be a good class of systems that we



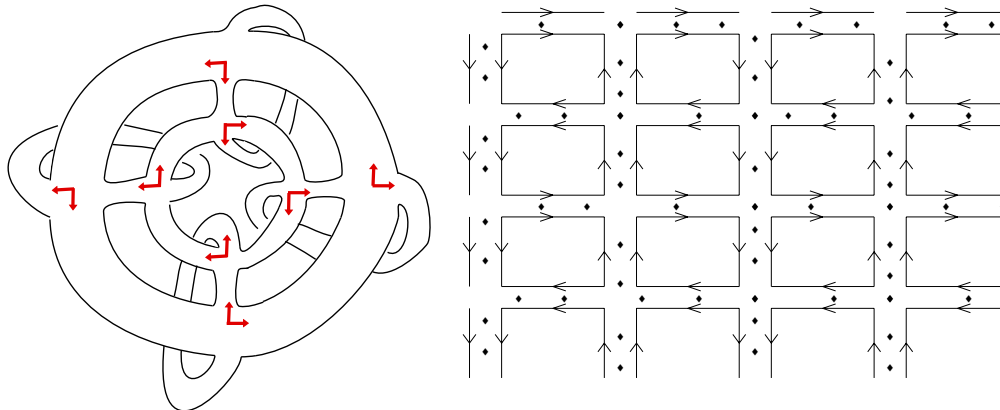


Fig. 11. Roads on a torus of  $4 \times 2$  streets with its authorized turn at junctions (left) and the asymptotic car distribution in the streets on a torus of  $4 \times 4$  streets obtained by simulation.

can describe by two matrices, one in the standard algebra and one in the minplus algebra. The standard compositions of these systems can be easily described in terms of these two matrices. These compositions of simple systems are useful to build the dynamics of large systems like that of regular town traffic.

In a forthcoming and more traffic-oriented paper, we will describe further the traffic interpretation of the four phases valid also for more general systems like regular towns. The influence on the fundamental diagram of the traffic control, using signal lights, will also be studied.

## VI. APPENDIX

*Lemma 1:* The eigenvalue problem (8) of size  $N$  can be reduced to the eigenvalue problem (9) of size 4.

*Proof:* First let us verify that the solutions of (8) verify  $\lambda \leq 1/4$ . Indeed (8) imply that  $\lambda q_{n+m} \leq \bar{a}_{n+m} q_1 q_{n+1} / (\lambda q_n)$ ,  $\lambda q_1 \leq a_n \sqrt{q_n q_{n+m}}$  and  $\lambda q_{n+1} \leq a_{n+m} \sqrt{q_n q_{n+m}}$ . Multiplying these three inequalities we obtain  $\lambda^4 \leq \bar{a}_{n+m} a_n a_{n+m} = 1$ .

To obtain the result we have to eliminate  $q_i, i \neq 1, n, n+1, n+m$ , that is to solve a linear minplus system which has a unique solution as soon as  $\lambda < 1/2$ . Indeed the loops of the precedence graph associated to the linear system has all its loops positive when  $\lambda \leq 1/2$ .

Moreover we can compute explicitly its solution. For  $i = 2, n - 1$  we have :

$$q_i = q_1 \left[ \bigotimes_{j=1}^{i-1} (a_j/\lambda) \right] \oplus \left[ \bigotimes_{j=i}^{n-1} (\bar{a}_j/\lambda) \right] q_n ,$$

for  $i = n + 1, n + m - 1$  :

$$q_i = q_{n+1} \left[ \bigotimes_{j=n+1}^{i-1} (a_j/\lambda) \right] \oplus \left[ \bigotimes_{j=i}^{n+m-1} (\bar{a}_j/\lambda) \right] q_{n+m} .$$

Using this explicit solution in the four last equations of (8) we obtain the reduced system (9). ■

To verify the results given Table I, let us rewrite the system (9) with simplified notations :  $U = q_n, V = q_{n+m}, X = q_1, Y = q_{n+1}, g = b_n, h = b_m, k = a_n, l = a_{n+m}, wn' = n - 1,$  and  $m' = m - 1.$

$$\begin{cases} U = \bar{k}XY/\lambda V \oplus gX/\lambda^{n'} , \\ V = \bar{k}XY/\lambda^2 U \oplus hY/\lambda^{m'} , \\ X = k\sqrt{UV}/\lambda \oplus \bar{g}U/\lambda^{n'} , \\ Y = l\sqrt{UV}/\lambda \oplus \bar{h}V/\lambda^{m'} , \end{cases} \quad (10)$$

with  $\bar{k} = 1 - k - l \geq 0, \bar{g} = n' - g \geq 0$  and  $\bar{h} = m' - h \geq 0$  are the free places in the crossing and the two roads.

*Lemma 2:* The eigenvector  $(U, V, X, Y)$  of the minplus nonlinear system (10) is given (using minplus notations) by :

	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta, \gamma) < d < \max(\beta, \gamma)$	$\gamma \leq d \leq 1$
$U$	$g/\lambda^{n'}$	$g/\lambda^{n'}$	$g/\lambda^{n'}$	$\bar{k}\bar{h}$
$V$	$\lambda^{n'+2}/k^2 g = hl/k\lambda^{m'}$	$1\lambda^{n'-2}/k^2 g = \lambda^{n'+2}/k^2 g$	$\lambda^{n'+2}/k^2 g$	$e/k^2 \bar{k}\bar{h}$
$X$	$e$	$e$	$e$	$e$
$Y$	$l/k$	$l/k$	$\lambda^{2+n'-m'} \bar{h}/k^2 g = \lambda^4 l/1k$	$e/k^2 \bar{k}$

(11)

where the eigenvalue  $\lambda$  of the minplus nonlinear system (10) has been given in Theorem 9  $\square$

*Proof:* Let us consider the four density regions corresponding to the four columns of this table :

1)  $0 \leq d \leq \alpha$  : the first column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = hY/\lambda^{m'}, \quad X = k\sqrt{UV}/\lambda, \quad Y = l\sqrt{UV}/\lambda, \quad (12)$$

which is itself a solution of System (10) since :

- a)  $\bar{k}XY/\lambda VU = \bar{k}kl/\lambda^3 = 1/\lambda^3 \geq 0$  (since  $\lambda \leq 1/4$  indeed, once more, using (10) we have  $VXY \leq \bar{k}klVXY/\lambda^4$ ),
- b)  $\bar{k}XY/\lambda^2 VU = 1/\lambda^4 \geq 0$ ,
- c)  $\bar{g}U/X\lambda^{n'} = (1/\lambda^2)^{n'} \geq 0$ ,
- d)  $\bar{h}V/Y\lambda^{m'} = \bar{h}h/\lambda^{m'} = (1/\lambda^2)^{m'} \geq 0$ .

Moreover, multiplying the 4 equalities of (12) we obtain  $\lambda^{n'+m'+2} = ghkk$  which gives the value of  $\lambda$  given in Table I.

- 2)  $\alpha \leq d \leq \beta$  : the second column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = \bar{k}XY/\lambda^2 U, \quad X = k\sqrt{UV}/\lambda, \quad Y = l\sqrt{UV}/\lambda, \quad (13)$$

which is itself solution of System (10) since :

- a)  $\bar{k}XY/\lambda VU = 1/\lambda^3 \geq 0$ ,
- b)  $hY/V\lambda^{m'} = hglk/\lambda^{n'+m'+2} = d^{m'+n'+1}/(mn)^{1/4} \geq 0$  since  $d \geq \alpha$ ,
- c)  $\bar{g}U/X\lambda^{n'} = \bar{g}g/\lambda^{2n'} = (1/\lambda^2)^{n'} \geq 0$ ,
- d)  $\bar{h}V/Y\lambda^{m'} = m'\lambda^{n'+2-m'}/hkg l = m'(2n'/m')^{1/4}/d^{m'+n'+1} \geq 0$  since  $d \leq \beta$ .

Moreover using the equality giving two expressions for the value of  $V$  in (11) we obtain  $\lambda = 1/4$ .

- 3)  $\min(\beta, \gamma) \leq d \leq \max(\beta, \gamma)$  : the third column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = \bar{k}XY/\lambda^2 U, \quad X = k\sqrt{UV}/\lambda, \quad Y = \bar{h}V/\lambda^{m'}, \quad (14)$$

which is itself solution of System (10) since :

- a)  $\bar{k}XY/\lambda VU = \lambda \geq 0$ ,
- b)  $hY/V\lambda^{m'} = h\bar{h}/\lambda^{2m'} = (1/\lambda^2)^{m'} \geq 0$  since  $\lambda \leq 1/4$ ,
- c)  $\bar{g}U/X\lambda^{n'} = \bar{g}g/\lambda^{2n'} = (1/\lambda^2)^{n'} \geq 0$ ,
- d)  $l\sqrt{UV}/\lambda Y = 1/\lambda^4 \geq 0$ .

Moreover using the equality giving two expressions for the value of  $Y$  in (11) we obtain  $\lambda^{2+n'-m'} m' = klgh = (m' + n' + 1)d$  which gives the value of  $\lambda$  given in Table I.

- 4)  $\gamma \leq d \leq 1$  : the fourth column of Table I is solution of the standard linear system :

$$U = \bar{k}XY/\lambda V, \quad V = \bar{k}XY/\lambda^2 U, \quad X = k\sqrt{UV}/\lambda, \quad Y = \bar{h}V/\lambda^{m'}, \quad (15)$$

which is itself solution of System (10) since :

- a)  $gX/\lambda^{n'}U = g/\bar{k}\bar{h} = ghkl/m = d^{n+m-1}/m \geq 0$  (since  $d \geq \gamma$ ) ,
- b)  $hY/\lambda^{m'}V = h\bar{h} = m' \geq 0$  ,
- c)  $\bar{g}U/X\lambda^{n'} = \bar{g}\bar{k}\bar{h} \geq 0$  ,
- d)  $l\sqrt{UV}/\lambda Y = lk\bar{k} = 1$  .

Moreover the compatibility of first two equalities of (15) implies that  $\lambda = 0$ .

■

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