Degree one homogeneous minplus dynamic systems and traffic applications : Part II

N. Farhi, M. Goursat & J.-P. Quadrat

Abstract. In this second part we discuss the phases appearing in the fundamental diagrams of traffic systems modeled by 1-homogeneous minplus dynamics and show the improvement obtained by traffic light control.

Mpreover, we have shown in the first part that 1-homogeneous systems may have a chaotic behavior, we give here a new subclass of 1-homogeneous dynamics having periodic trajectories. It generalizes the standard cases which need a monotony property. Moreover we show that this new, but still restrictive class, has applications to regular town traffic with crossings but without turning possibilities.

1. The traffic fundamental diagram phases.

The fundamental diagrams of quite different systems are similar to the one given in part I. We have studied the cases of two circular roads with one crossing and two crossings and the cases of regular towns with various number of roads on a torus. In all these cases we suppose the existence of right priority.

The fundamental diagrams have always three phases corresponding respectively to low, average and high densities. We see on the fundamental diagram of Part I that : – for low densities the flow increases linearly with the density, – for average densities the flow is constant, – for high density there are deadlocks and the flows are null.

On Figures (1), (2) and (3) we show the typic asymptotic distribution of vehicles in the three phases [**3, 4**].

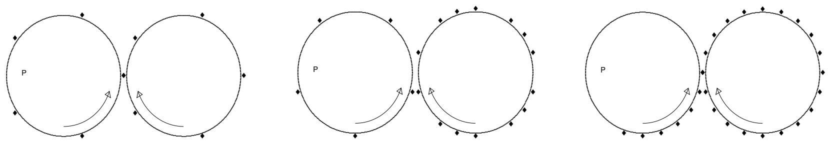

Figure 1. Two circular roads with one crossing case. Car distributions in the low average and high density phases.

We see that :

Key words and phrases. Traffic modeling, Petri nets, Minplus algebra.
Partially supported by RFBR/CNRS grant 05-01-02807.

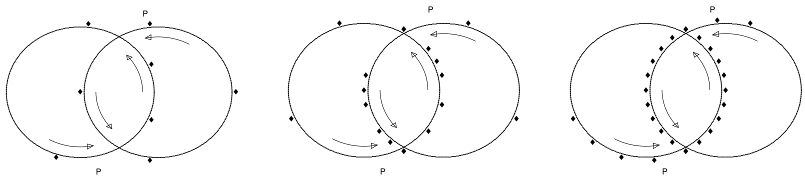

FIGURE 2. Four roads with two crossings. Car distributions in the low average and high density phases.

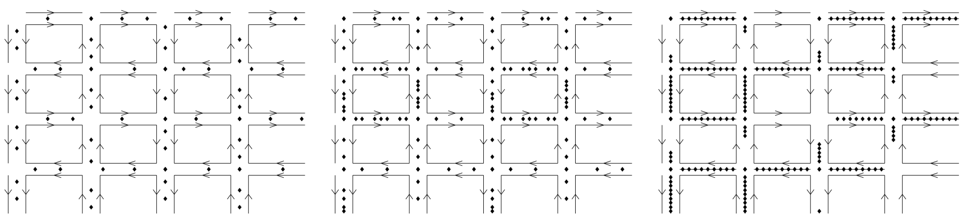

FIGURE 3. A regular town. Car distributions in the low average and high density phases.

- *Low density phase.* There are so few vehicles in the network that after a transient regime, they move without obstructing each other on the roads and in the crossings. Thus, the "priority to the right" is not used, the vehicles moves as on a unique circular road and the average flow is equal to the vehicle density in the network. This phase corresponds to densities less than 1/4.
- *Average density phase.* When the density is between 1/4 and 1/2 (in the symmetric road cases), the vehicles can neither move freely on the roads, nor avoid each other on the crossings. Therefore "priority to the right" happens. The car on the priority road move freely and the waiting cars are all in the non priority road. The flow reaches the maximum value 1/4 corresponding to the full use of the crossings.
- *High density phase.* When the density exceeds a quantity equal to 1/2 (in the symmetric case), at asymptotic regime, a closed circuit of vehicles on some nonpriority roads appears which creates a complete deadlock of the system.

2. Traffic light control.

To avoid the deadlock due to right priority we can use traffic light controls. A Petri net describing the junction with the traffic light control is shown on Figure 4. The negative weight extension of Petri net is necessary to model the light phases in a time invariant way. The part of the Petri modeling the light control corresponds to the places $a_g, a_c, \bar{a}_g, \bar{a}_c$. As long as a_c contains a token $a_c = 1$ the green light is for the North street, when $\bar{a}_c = 1$ the green light is for the East street. As long as $a_c = 1$ we have $a_g = 1$ and q_v is authorized to fire (since thanks to the loop q_v, a_g, q_v as soon as a token is consumed another one is generated in the place a_g). The main point is that when the token in a_c goes in $\bar{a}_c$ (phase change) the tokens in a_g must be removed (this is done by the input arc with weight -1 of the place a_g). More generally without negative weight we cannot model tokens staying less then a prescribed time.

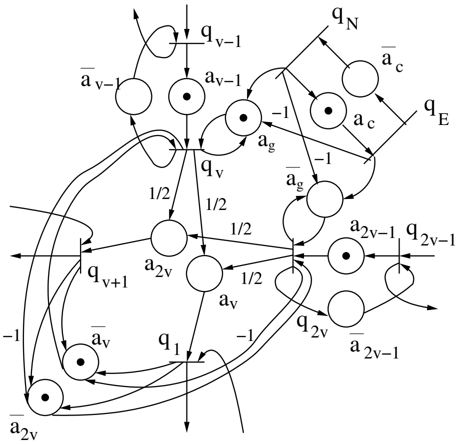

FIGURE 4. Traffic lights modeling.

In Figure 5, we compare the fundamental diagrams of three crossing policies for a system composed of two circular roads of same size with two junctions. The three policies are : – right priority, – standard given phase duration, – feedback controlled duration (based on the road congestion) computed by LQG method.

The control improves the average and the high density phases, without spoiling the low density one. The improvement given by the feedback control achieves the throughput obtained on a unique circular road without crossing but doubling the time spent in a place representing the crossing place.

Furthermore, the feedback control dissolves more efficiently the jams (that can appear locally in transient regimes) than the other policies would do.

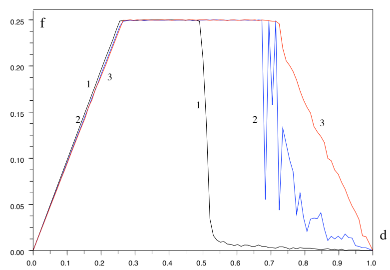

FIGURE 5. Comparison of three policies of managing the crossings : –right priority to the right (1), –open loop light control (2), –feedback light control (3).

3. A subclass of triangular homogeneous dynamics

In this section we study a subclass of 1-homogeneous minplus linear systems for which we can prove the periodicity. Their dynamics belongs to a subclass of 1-homogeneous triangular systems :

$$\begin{cases} u_{k+1} = C \otimes u_k, \\ x_{k+1} = A(u_k) \otimes x_k \oplus B(u_k) \otimes u_k. \end{cases} \tag{3.1}$$

where $\{u_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ are minplus column vectors, C is a minplus square matrix, $A(u_k)$ and $B(u_k)$ are two minplus 0-homegeneous matrices depending of u_k. We call this type of systems Triangular 1-Homogeneous (T1H).

We call linear periodic dynamic (LP) a dynamic given by :

$$x_{k+1} = A_k \otimes x_k, \quad x_0 \text{ given},$$

where A_k are minplus matrices periodic in k.

We can prove the following theorems (see the proves in [**5**]).

THEOREM 3.1. *Every T1H dynamics behaves asymptotically as a LP dynamics.*

THEOREM 3.2. *A T1H system with $A(u)$ irreducible for every $u \in \mathbb{R}_{min}$ satisfies :*

$$\max_{u_0\in\mathbb{R}_{\min}} \mu_x(u_0) = \max_{\bar{u}\in\mathcal{V}} \mu_x(\bar{u}),$$

where : – u_0 denotes the initial condition of u, – $\mu_x(u_0) = \lim_{k\to\infty} x_k/k$, – $\mathcal{V}$ is the set of the minplus eigen vectors of the matrix C.

THEOREM 3.3. *Every LP dynamic $y_{k+1} = E_k \otimes y_k$, such that the matrices E_k have the same support, is realizable by a T1H dynamics.*

4. Application to traffic

We show that the traffic of regular towns with traffic light, buffered junction but without turning possibilities can be modeled with a T1H dynamics.

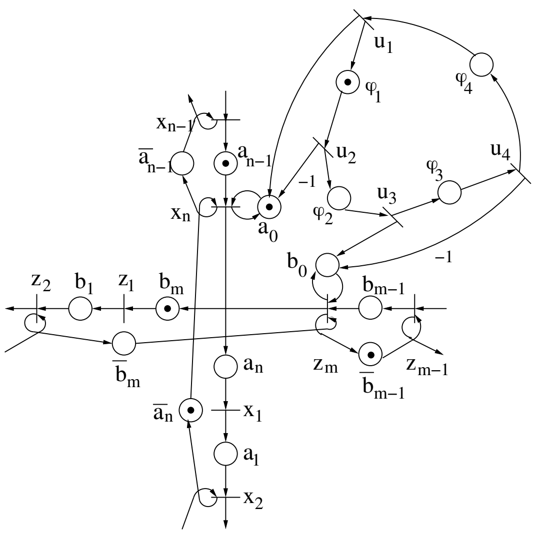

FIGURE 6. Traffic light intersection without possibility of turning.

On the Petri net of Figure 6 the traffic light is modeled by the subsystem corresponding to the transitions u_1, u_2, u_3, u_4, which has no input coming from the rest of the system. The dynamic of this subsystem is minplus linear. If the initial condition $u_0 = (0, 0, 0, 0)$ the number of tokens in the places a_0 and b_0 is boolean and periodic. To a cycle corresponds the four phases given in the Table 1. The junction has a buffer place in each direction

Phase	a_0	b_0	Vertical light color	Horizontal light color
1	1	0	green	red
2	0	0	red	red
3	0	1	red	green
4	0	0	red	red

TABLE 1. The phases of the traffic light.

(a_1, b_1) to avoid blocking. The phases 2 and 4 gives the time, for car entering in the junction, to go in the buffer and then to free the crossing. Indeed, a vehicle entering in the crossing (represented by the two places a_n and b_m) leaves it surely in one unit of time.

The green duration of phase 1 and 3 is the sojourn time of tokens in the place φ_i. The phases 2 and 4 have a duration of one unit.

PROPOSITION 4.1. *The dynamics of the Petri net of Figure 6 has the T1H dynamics :*

$$u^{k+1} = \begin{bmatrix} \cdot & \cdot & \cdot & \varphi_4 \\ \varphi_1 & \cdot & \cdot & \cdot \\ \cdot & \varphi_2 & \cdot & \cdot \\ \cdot & \cdot & \varphi_3 & \cdot \end{bmatrix} \otimes u^k, \qquad \begin{bmatrix} x^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} A_1(u^k) & \cdot \\ \cdot & A_2(u^k) \end{bmatrix} \otimes \begin{bmatrix} x^k \\ z^k \end{bmatrix},$$

where $\cdot$ denotes ∞, with

$$A_1(u)_{i,j} = \begin{cases} a_0 u_1/u_2 \text{ if } (i,j) = (n,n), \\ \text{independent of } u \text{ elsewhere.} \end{cases}$$

and

$$A_2(u)_{i,j} = \begin{cases} b_0 u_3/u_4 \text{ if } (i,j) = (m,m), \\ \text{independent of } u \text{ elsewhere.} \end{cases}$$

We are able to explicit the asymptotic flows which are different according the direction followed by the vehicles.

THEOREM 4.2. *The average flow on the horizontal (resp. vertical) road is given by $\lambda/4$ where λ is the unique eigenvalue of the irreducible matrix $\bigotimes_{k=0}^{3} A_1(u^k)$ [resp. $\bigotimes_{k=0}^{3} A_2(u^k)$].*

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat : *Synchronization and Linearity*, Wiley, 1992.

[2] M. Gondran, M. Minoux : *Graphs and Algorithms*, J. Wiley & Sons, 1986.

[3] N. Farhi, M. Goursat, and J.P. Quadrat : *Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling*, In Proceedings of the 44th IEEE - CDC-ECC Seville December 2005.
[4] N. Farhi, M. Goursat, and J.P. Quadrat: *Fundamental Traffic Diagrams of Elementary Road Networks*, to appear in Proceedings of ECC Kos July 2007.
[5] N. Farhi : *A class of periodic degree one homogeneous minplus systems*, to appear.
[6] G. Cohen, S. Gaubert and J.P. Quadrat : *Asymptotic Throughput of Continuous Timed Petri Nets* Proceedings of the 34th Conference on Decision and Control, New Orleans, Dec, 1995.
[7] S. Gaubert : *Théorie des systémes linéaires dans les dioides*, Thesis dissertation, École des Mines de Paris, 1992.

INRIA - Rocquencourt, B.P. 105, 78153, Le Chesnay, Cedex France.
E-mail address: `Nadir.Farhi@inria.fr`