SEMIRING, PROBABILITY AND DYNAMIC PROGRAMMING

MAX-PLUS WORKING GROUP

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This work has been partly supported by the ALAPEDES project of the European TMR programme.

Contents

1. Structures

- A semiring $\mathcal{K}$ is a set endowed with two operations denoted $\oplus$ and $\otimes$ where $\oplus$ is associative, commutative with zero element denoted ε, $\otimes$ is associative, admits a unit element denoted e, and distributes over $\oplus$; zero is absorbing ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$ for all $a \in \mathcal{K}$). This semiring is commutative when $\otimes$ is commutative.
- A module on a semiring is called a semimodule.
- A dioid $\mathcal{K}$ is a semiring which is idempotent ($a \oplus a = a, \ \forall a \in \mathcal{K}$).
- A [commutative, resp. idempotent] semifield is a [commutative, resp. idempotent] semiring whose nonzero elements are invertible.
- We denote $\mathcal{M}_{np}(\mathcal{K})$ the semimodule of (n, p)-matrices with entries in the semiring $\mathcal{K}$. When $n = p$, we write $\mathcal{M}_n(\mathcal{K})$. It is a semiring with matrix product :

$$[AB]_{ij} \stackrel{\text{def}}{=} [A \otimes B]_{ij} \stackrel{\text{def}}{=} \bigoplus_k [A_{ik} \otimes B_{kj}] \ .$$

All the entries of the zero matrix are ϵ. The diagonal entries of the identity matrix are e, the other entries being ϵ.

1.1. EXAMPLES OF SEMIRING

$\mathcal{K}$	$\oplus$	$\otimes$	ε	e	name
$\mathbb{R}^+$	$+$	$\times$	0	1	$\mathbb{R}^+$
$\mathbb{R}^+$	$\sqrt[p]{a^p + b^p}$	$\times$	0	1	$\mathbb{R}_p^+$
$\mathbb{R}^+$	max	$+$	0	1	$\mathbb{R}_{\max,\times}$
$\mathbb{R} \cup \{+\infty\}$	min	$+$	$+\infty$	0	$\mathbb{R}_{\min}$
$\mathbb{R} \cup \{-\infty, +\infty\}$	min	$+$	$+\infty$	0	$\overline{\mathbb{R}}_{\min}$
$\mathbb{R} \cup \dot{\mathbb{R}}$	a max$(\lvert a\rvert, \lvert b\rvert)$	$\times$	0	1	$\mathbb{S}$
$[a, b]$	max	min	b	a	$[a, b]_{\max,\min}$
$\{0, 1\}$	and	or	0	1	$\mathbb{B}$
$\mathcal{P}(\Sigma^*)$	$\cup$	prod. lat.	$\emptyset$	$-$	$\mathbb{L}$

In $\mathbb{S}$ we have $\ominus 2 \triangleq \oplus - 2$; $\dot{2} = 2 \ominus 2 = (2, -2)$; $3 \ominus 2 = 3$;
$-3 \oplus 2 = -3$; $\dot{2} \oplus 3 = 3$; $\dot{2} \ominus 3 = -3$; $\dot{2} \oplus 1 = \dot{2} \ominus 1 = \dot{2}$.

1.2. Matrices and Graphs

- With a matrix C in $\mathcal{M}_n(\mathcal{K})$, we associate a precedence graph $\mathcal{G}(C) = (\mathcal{N}, \mathcal{P})$ with nodes $\mathcal{N} = \{1, 2, \cdots, n\}$, and arcs $\mathcal{P} = \{xy \mid x, y \in \mathcal{N},\ C_{xy} \neq \varepsilon\}$.
- The weight of a path π, denoted $\pi(C)$, is the $\otimes$-product of the weights of its arcs. For example we have $xyz(C) = C_{xy} \otimes C_{yz}$.
- The length of the path π (is $\pi(1)$ when $\otimes$ is $+$ (its weight when the arc weigths are all equal to 1)).
- The set of all paths with ends xy and length l is denoted $\mathcal{P}^l_{xy}$. Then, $\mathcal{P}^*_{xy}$ is the set of all paths with ends xy and $\mathcal{P}^*$ the set of all paths.

$$\mathcal{P}^* \stackrel{\text{def}}{=} \bigcup_{l=0}^{\infty} \mathcal{P}^l . \ \ \mathcal{C} = \bigcup_{x} \mathcal{P}^*_{xx} . \ \ \rho \subset \mathcal{P}^*, \ \ \rho(C) \stackrel{\text{def}}{=} \bigoplus_{\pi \in \rho} \pi(C) \ .$$

- We define the star operation by $C^* \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} C^i$.

PROPOSITION 1. *For* $C \in \mathcal{M}_n(\mathcal{K})$ *we have*

$$\mathcal{P}^l_{xy}(C) = C^l_{xy}, \; \mathcal{P}^*_{xy}(C) = C^*_{xy} \; . \tag{1}$$

- If $\mathcal{K} = \mathbb{R}^+$ and $Ce = e$, the equation $p^{n+1} = p^n C$ is the forward Kolmogorov equation.
- If $\mathcal{K} = \mathbb{R}^+$ and $Ce = e$, C^*_{xy} is the probability to reach y starting from x.
- If $\mathcal{K} = \mathbb{R}_{\min}$, the equation $v^{n+1} = v^n C$ is the forward dynamic programming equation.
- If $\mathcal{K} = \mathbb{R}_{\min}$, the eigen equation $\lambda v = vC$ is the ergodic (average cost by unit of time) dynamic programming equation.
- If $\mathcal{K} = \mathbb{R}_{\min}$ and C irreducible, C admits a unique eigenvalue λ, $\lambda = \bigoplus_{\pi \in \mathcal{C}} \frac{\pi(C)}{\pi(1)}$, the columns $\{(C/\lambda)^+_{.x} \mid (C/\lambda)^+_{xx} = e\}$ with $C^+ = CC^*$ generate the corresponding eigensemidodule.
- If $\mathcal{K} = \mathbb{R}_{\min}$ and $\lambda \geq e$, $C^* = e \oplus C \cdots C^{n-1}$ and C^*_{xy} is the minimal weight of the paths joining x to y which is finite.

1.3. COMBINATORICS - CRAMER FORMULAS

THEOREM 2. *The solution of the system* $Ax \oplus b' = A'x \oplus b$ *in* $\mathbb{R}^{+}_{\max,\times}$ *exists and is unique and given by*[1]

$$x = (A \ominus A')^{\sharp}(b \ominus b')/\det\left(A \ominus A'\right) ,$$

$$\det(A) = \bigoplus_{\sigma} sgn(\sigma) \bigotimes_{i=1}^{n} A_{i\sigma(i)} , \quad A^{\sharp}_{ij} = cofactor_{ji}(A) ,$$

when and only when $x \geq 0$.

$$\begin{cases} \max(x_1, 3x_2) = 5, \\ \max(4x_1, 2x_2) = 6, \end{cases} \qquad \det(A) = 2 \ominus 12 = \ominus 12, \quad \det \begin{bmatrix} 5 & 3 \\ 6 & 2 \end{bmatrix} = \ominus 18,$$

$$\det \begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix} = \ominus 20, \quad x_1 = 3/2, \; x_2 = 5/3, \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} .$$

[1] The computation are done in $\mathbb{S}$.

1.4. Order - Residuation

- A dioid is complete when the $\otimes$ is distributive with the infinite $\oplus$.
- A complete dioid is a lattice ($\oplus$ upper bound, $\wedge$ lower bound).
- $\mathcal{D}$ and $\mathcal{C}$ complete dioids $f : \mathcal{D} \to \mathcal{C}$. f is residuable if $\{x \mid f(x) \leq y\}$ admits an maximal element denoted by $f^{\sharp}(y)$.
- f residuable $\Leftrightarrow$ $f \circ f^{\sharp} \leqslant I_{\mathcal{C}}$ and $f^{\sharp} \circ f \geqslant I_{\mathcal{D}}$.

1. $f \circ f^{\sharp} \circ f = f$. $\quad f^{\sharp} \circ f \circ f^{\sharp} = f^{\sharp}$.
2. f is injective $\Longleftrightarrow$ $f^{\sharp} \circ f = I_{\mathcal{D}} \Longleftrightarrow$ $f^{\sharp}$ is surjective and the dual.
3. $(h \circ f)^{\sharp} = f^{\sharp} \circ h^{\sharp}$. $\quad f \leqslant g \Longleftrightarrow g^{\sharp} \leqslant f^{\sharp}$.
4. $(f \oplus g)^{\sharp} = f^{\sharp} \wedge g^{\sharp}$. $\quad (f \wedge g)^{\sharp} \geqslant f^{\sharp} \oplus g^{\sharp}$.

In $\mathbb{R}_{\max}$ if $f(x) = Ax$ then $f^{\sharp}(y)_j = (A \backslash y)_j \triangleq \bigwedge_i y_i / A_{ij}$.

X and Y semodules, $F : X \to Y$ a linear map.

- $\mathrm{Im}(F) = \{F(x) \mid x \in X\}$.
- $\ker(F) = \left\{\left(x^1, x^2\right) \in X^2 \mid F\left(x^1\right) = F\left(x^2\right)\right\}$. It is a congruence that is an equivalent relation $\mathcal{R} \subset X \times X$ which is a semimodule.

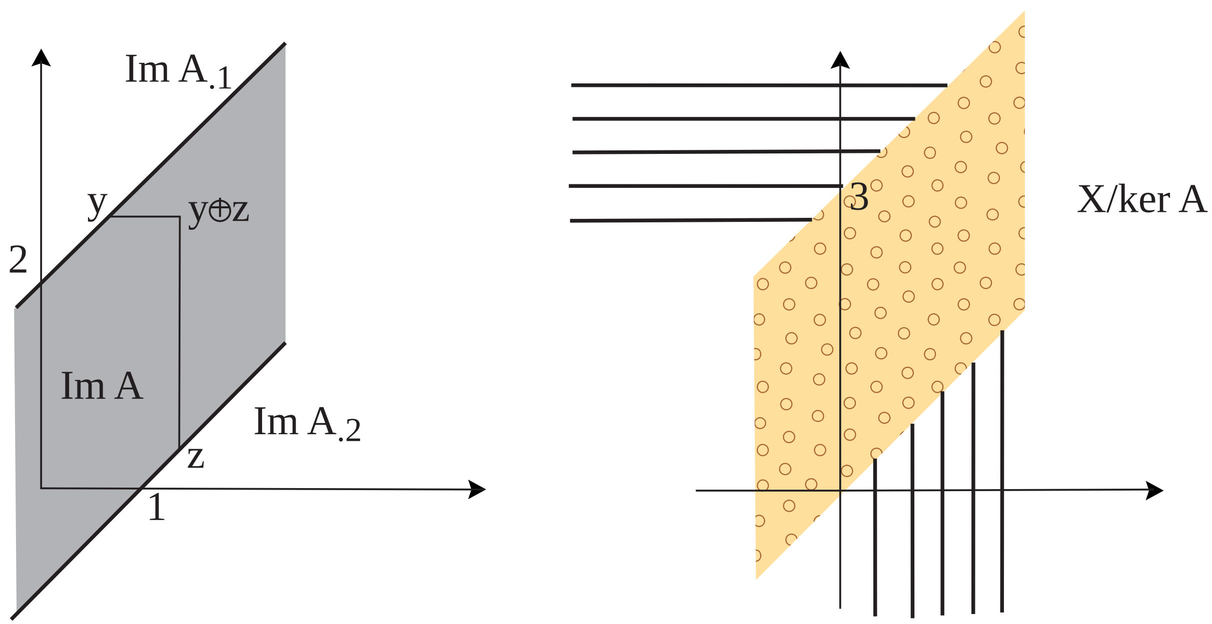

FIGURE 1. Image and Kernel.

- A generating family $\{x_i\}_{i\in I}$ of a semimodule X is a subset of X :

$$\forall x \in X \;\; \exists \{\alpha_i\}_{i\in I} \in \mathcal{K} : \; x = \bigoplus_{i\in I} \alpha_i x_i .$$

- "Convex" semimodule admits a unique generating family (the set of the extremal points).
- The family $\{x_i\}_{i\in I}$ is independent if

$$\bigoplus_{i\in I} \alpha_i x_i = \bigoplus_{i\in I} \beta_i x_i \Longrightarrow \alpha_i = \beta_i , \quad \forall i \in I \; .$$

- An independent generating family is called a basis. A semimodule admitting a basis is called free.

$$p_1 = \begin{bmatrix} \varepsilon \\ e \\ e \end{bmatrix}, \;\; p_2 = \begin{bmatrix} e \\ \varepsilon \\ e \end{bmatrix}, \;\; p_3 = \begin{bmatrix} e \\ e \\ \varepsilon \end{bmatrix}, \quad p_1 \oplus p_2 = p_2 \oplus p_3 \; .$$

1.6. Regular Matrices and Projective Semimodules

- A matrix A is regular if it exists a matrix $A^\sharp : AA^\sharp A = A$.
- A subsemimodule V of a semimodule E and a congruence $\mathcal{R}$ of E form a direct sum $E \triangleq V \boxplus \mathcal{R}$ if

$$\forall x \in E \quad \exists ! y \in V \ : \ x\mathcal{R}y \ .$$

  y is called the projection of x on V parallel to $\mathcal{R}$.
- A semimodule V is said projective if it exists $\mathcal{R}$ congruence and E a free semimodule such that $E = V \boxplus \mathcal{R}$.

Theorem 3. *Given* $A = \mathcal{M}_n(\mathbb{R}_{\max})$, $\mathrm{Im}(A)$ *is projective iff* A *is regular then it exists* B *with* $E = \mathrm{Im}(A) \boxplus \ker B$ *and* $P \triangleq A(BA\backslash B)$ *is the linear projector on* $\mathrm{Im}(A)$ *parallel to* $\ker(B)$.

2. Cost Measures and Decision Variables

We call a decision space the triplet $(U, \mathcal{U}, \mathbb{K})$ where U is a topological space, $\mathcal{U}$ the set of open sets of U and $\mathbb{K}$ a mapping from $\mathcal{U}$ to $\mathbb{R}_{\min}$ such that

1. $\mathbb{K}(U) = 0$,
2. $\mathbb{K}(\emptyset) = +\infty$,
3. $\mathbb{K}\left(\bigcup_n A_n\right) = \inf_n \mathbb{K}(A_n)$ for any $A_n \in \mathcal{U}$.

The mapping $\mathbb{K}$ is called a cost measure.

A set of cost measures K is said tight if

$$\sup_{C \text{compact} \subset U} \inf_{\mathbb{K} \in K} \mathbb{K}(C^c) = +\infty .$$

A mapping $c : U \to \mathbb{R}_{\min}$ such that $\mathbb{K}(A) = \inf_{u \in A} c(u) \ \forall A \subset U$ is called a cost density of the cost measure $\mathbb{K}$.

THEOREM 4 (M. Akian, V.N. Kolokoltsov). *Given a l.s.c.* c *with values in* $\mathbb{R}_{\min}$ *such that* $\inf_u c(u) = 0$, *the mapping* $A \in \mathcal{U} \mapsto \mathbb{K}(A) = \inf_{u \in A} c(u)$ *defines a cost measure on* $(U, \mathcal{U})$.

Conversely any cost measure defined on a topological space with a countable basis of open sets admits a unique minimal extension $\mathbb{K}_*$ *to* $\mathcal{P}(U)$ *(the set of subsets of* U*) having a density* c *which is a l.s.c. function on* U *satisfying* $\inf_u c(u) = 0$.

EXAMPLE 5. 1. $\chi_m(x) \stackrel{\text{def}}{=} \begin{cases} +\infty & \text{for } x \neq m. \\ 0 & \text{for } x = m, \end{cases}$

2. $\mathcal{M}^p_{m,\sigma}(x) \stackrel{\text{def}}{=} \frac{1}{p} \|\sigma^{-1}(x - m)\|^p$ for $p \geq 1$ with $\mathcal{M}^p_{m,0} \stackrel{\text{def}}{=} \chi_m$.

By analogy with the conditional probability we define conditional cost excess to take the best decision in A knowing that it must be taken in B by

$$\mathbb{K}(A|B) \stackrel{\text{def}}{=} \mathbb{K}(A \cap B) - \mathbb{K}(B) .$$

2.1. Decision Variables

1. A decision variable X on $(U, \mathcal{U}, \mathbb{K})$ is a mapping from U to E (a second countable topological space). It induces a cost measure $\mathbb{K}_X$ on $(E, \mathcal{B})$ ($\mathcal{B}$ denotes the set of open sets of E) defined by

$$\mathbb{K}_X(A) = \mathbb{K}_*(X^{-1}(A)), \ \forall A \in \mathcal{B} .$$

   The cost measure $\mathbb{K}_X$ has a l.s.c. density denoted c_X .
2. Two decision variables X and Y are said independent when:

$$c_{X,Y}(x, y) = c_X(x) + c_Y(y).$$

3. The conditional cost excess of X knowing Y is defined by:

$$c_{X|Y}(x, y) \stackrel{\text{def}}{=} \mathbb{K}_*(X = x \mid Y = y) = c_{X,Y}(x, y) - c_Y(y).$$

4. The optimum of a decision variable is defined by

$$\mathbb{O}(X) \stackrel{\text{def}}{=} \arg\min_{x \in E} \operatorname{conv}(c_X)(x)$$

5. When the optimum of a decision variable X with values in $\mathbb{R}^n$ is unique and when near the optimum, we have

$$\mathrm{conv}(c_X)(x) = \frac{1}{p}\|\sigma^{-1}(x - \mathbb{O}(X))\|^p + o(\|x - \mathbb{O}(X)\|^p) \ ,$$

we say that X is of order p and we define its sensitivity of order p by

$$\mathbb{S}^p(X) \stackrel{\mathrm{def}}{=} \sigma \ .$$

6. The value[resp. conditional value] of a cost variable X is

$$\mathbb{V}(X) \stackrel{\mathrm{def}}{=} \inf_x (x + c_X(x)) \ \ , \mathbb{V}(X \mid Y = y) \stackrel{\mathrm{def}}{=} \inf_x (x + c_{X|Y}(x, y)) \ .$$

7. The cost densityof the sum Z of two independent variables X and Y is the inf-convolution of their cost densities c_X and c_Y, denoted $c_X \star c_Y$ defined by

$$c_Z(z) = \inf_{x,y} \left[c_X(x) + c_Y(y) \mid x + y = z \right] .$$

For a real decision variable X of cost $\mathcal{M}^p_{m,\sigma}$, $p > 1$, we have

$$\mathbb{O}(X) = m,\ \mathbb{S}^p(X) = \sigma,\ \mathbb{V}(X) = m - \frac{1}{p'}\sigma^{p'}\ .$$

THEOREM 6. *For $p > 0$, the numbers*

$$|X|_p \stackrel{\text{def}}{=} \inf\left\{\sigma \mid c_X(x) \geq \frac{1}{p}|(x - \mathbb{O}(X))/\sigma|^p\right\} \text{ and } \|X\|_p \stackrel{\text{def}}{=} |X|_p + |\mathbb{O}(X)|$$

define respectively a seminorm and a norm on the vector space $\mathbb{L}^p$ of real decision variables having a unique optimum and such that $\|X\|_p$ is finite.

THEOREM 7. *For two independent real decision variables X and Y and $k \in \mathbb{R}$ we have (as soon as the right and left hand sides exist)*

$$\mathbb{O}(X+Y) = \mathbb{O}(X) + \mathbb{O}(Y),\ \ \mathbb{O}(kX) = k\mathbb{O}(X),\ \ \mathbb{S}^p(kX) = |k|\mathbb{S}^p(X)\ ,$$

$$[\mathbb{S}^p(X+Y)]^{p'} = [\mathbb{S}^p(X)]^{p'} + [\mathbb{S}^p(Y)]^{p'},\ \ (|X+Y|_p)^{p'} \leq (|X|_p)^{p'} + (|Y|_p)^{p'}\ .$$

2.2. Characteristic Functions, Fenchel & Cramer Transform

- The Fenchel transform $\mathcal{F}$ of a convex function

$$\hat{c}(\theta) \stackrel{\text{def}}{=} [\mathcal{F}(c)](\theta) \stackrel{\text{def}}{=} \sup_x [\langle \theta, x \rangle - c(x)] \ .$$

- The characteristic function of a decision variable is defined by

$$\mathbb{F}(X) \stackrel{\text{def}}{=} \mathcal{F}(c_X) \ .$$

$$\mathbb{F}(X+Y) = \mathbb{F}(X) + \mathbb{F}(Y), \quad [\mathbb{F}(kX)](\theta) = [\mathbb{F}(X)](k\theta) \ .$$

- The Cramér transform $\mathcal{C}_r \stackrel{\text{def}}{=} \mathcal{F} \circ \log \circ \mathcal{L}$ associates to the probability law μ the convex function

$$c_\mu : U \mapsto \sup_\theta [\theta U - \log \mathbb{E}_\mu(e^{\theta\lambda})] \ ,$$

where $\mathcal{L}$ is the Laplace transform.

$\mathcal{M}$	$\log(\mathcal{L}(\mathcal{M})) = \mathcal{F}(\mathcal{C}(\mathcal{M}))$	$\mathcal{C}(\mathcal{M})$
μ	$\hat{c}_\mu(\theta) = \log \int e^{\theta x} d\mu(x)$	$c_\mu(x) = \sup_\theta(\theta x - \hat{c}(\theta))$
0	$-\infty$	$+\infty$
δ_a	θa	χ_a
Gauss distrib.	$m\theta + \frac{1}{2}\lvert\sigma\theta\rvert^2$	$\mathcal{M}^2_{m,\sigma}$
$\mu * \nu$	$\hat{c}_\mu + \hat{c}_\nu$	$c_\mu \star c_\nu$
$k\mu$	$\log(k) + \hat{c}$	$c - \log(k)$
$\mu \geq 0$	$\hat{c}$ convex l.s.c.	c convex l.s.c.
$m_0 \overset{\text{def}}{=} \int \mu$	$\hat{c}(0) = \log(m_0)$	$\inf_x c(x) = -\log(m_0)$
$m_0 = 1$	$\hat{c}(0) = 0$	$\inf_x c(x) = 0$
$m_0 = 1,\ m \overset{\text{def}}{=} \int x\mu$	$\hat{c}'(0) = m$	$c(m) = 0$
$m_0 = 1,\ m_2 \overset{\text{def}}{=} \int x^2\mu$	$\hat{c}''(0) = \sigma^2 \overset{\text{def}}{=} m_2 - m^2$	$c''(m) = 1/\sigma^2$

TABLE 1. Properties of the Cramer transform.

2.3. Convergences of Decision Variables

For the sequence of real decision variables $\{X_n, n \in \mathbb{N}\}$, cost measures $\mathbb{K}_n$ and c_n functions from U (a first countable topological space[2]) to $\mathbb{R}_{\min}$ we say that :

1. $X_n \in \mathbb{L}^p$ converges in p-norm towards $X \in \mathbb{L}^p$ denoted $X_n \xrightarrow{\mathbb{L}^p} X$, if $\lim_n \|X_n - X\|_p = 0$;
2. $\mathbb{K}_n$converges weakly towards $\mathbb{K}$, denoted $\mathbb{K}_n \xrightarrow{w} \mathbb{K}$, if for all f in $\mathcal{C}_b(E)$ [3] we have $\lim_n \mathbb{K}_n(f) = \mathbb{K}(f)$[4].

A sequence $\mathbb{K}_n$ of cost measures is said asymptoticaly tight if

$$\sup_{C \text{compact} \subset U} \liminf_n \mathbb{K}_n(C^c) = +\infty \ .$$

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[2] Each point admits a countable basis of neighbourhoods.

[3] $\mathcal{C}_b(E)$ denotes the set of continuous and lower bounded functions from E to $\mathbb{R}_{\min}$.

[4] $\mathbb{K}(f) \stackrel{\text{def}}{=} \inf_u(f(u) + c(u))$ where c is the density of $\mathbb{K}$.

THEOREM 8 (Large Numbers). *Given a sequence* $\{X_n,\ n \in \mathbb{N}\}$ *of i.i.c. decision variables belonging to* $\mathbb{L}^p,\ \ p \geq 1$, *we have*

$$Y_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=0}^{N-1} X_n \to \mathbb{O}(X_0) \ ,$$

where the limit is in p-norm convergence.

THEOREM 9 (Central Limit). *Given an i.i.c. sequence* $\{X_n, n \in \mathbb{N}\}$ *centered of order p with l.s.c. convex cost, we have*

$$Z_N \stackrel{\text{def}}{=} \frac{1}{N^{1/p'}} \sum_{n=0}^{N-1} X_n \stackrel{w}{\to} \mathcal{M}^p_{0,\mathbb{S}^p(X_0)} \ .$$

THEOREM 10 (Large Deviation). *Given an i.i.c. sequence* $\{X_n, n \in \mathbb{N}\}$ *of tight cost density* c, *we have :*

$$\frac{1}{n} c_{(X_1+\cdots+X_n)/n} \stackrel{w}{\to} \hat{c} \ ,$$

where $\hat{c}$ *denotes the convex hull of* c.

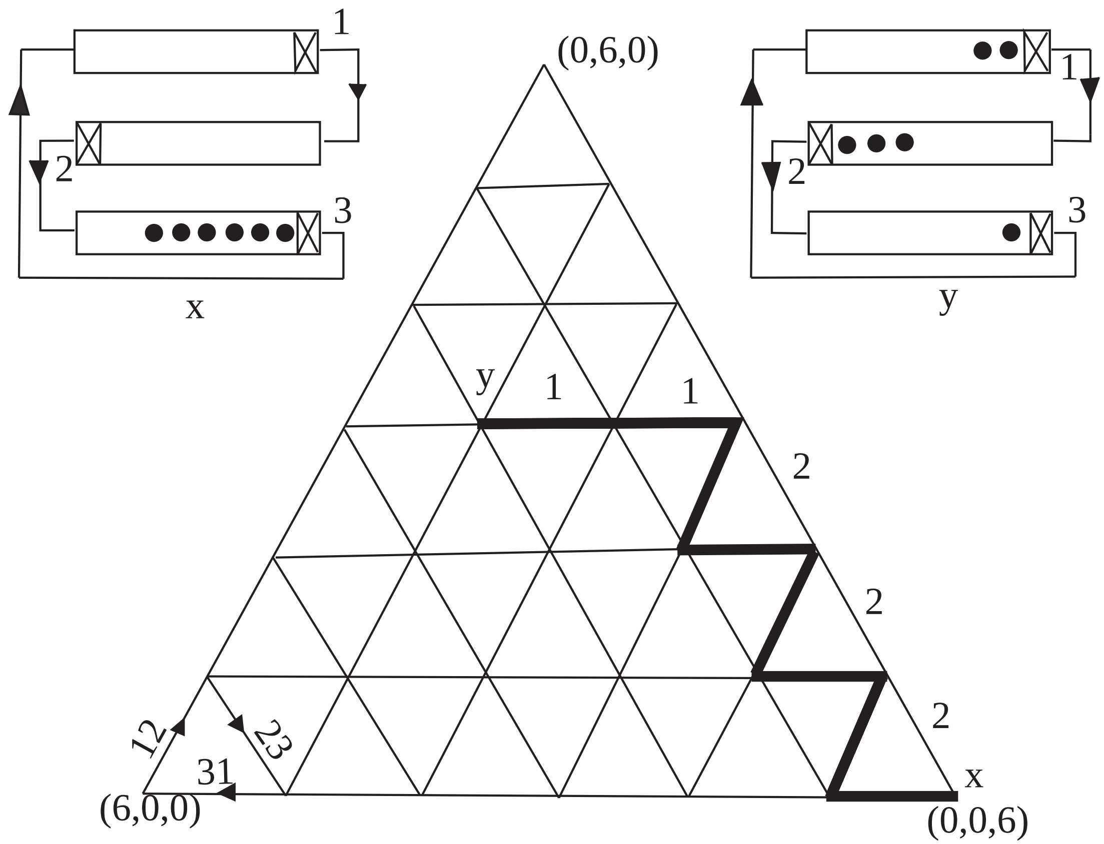

FIGURE 2. Transportation System (6 cars, 3 parkings).

- We consider a company renting cars Figure (2). It has n cars and m parkings in which customers can rent cars.
- The customers can rent a car in a parking and leave the rented car in another parking.
- After some time the distribution of the cars in the parkings is not satisfactory and the company has to transport the cars to achieve a better distribution.
- Given r the (m, m) matrix of transportation cost from a parking to another, the problem is to determine the minimal cost of the transportation from a distribution $x = (x_1, \cdots, x_m)$ of the cars in the parking to another one $y = (y_1, \cdots, y_m)$ and to compute the best plan of transportation.

3.1. PRECISE FORMULATION

- Given the (m, m) transition cost matrix r irreducible such that $r_{ij} > 0$ if $i \neq j = 1, \cdots, m$ and $r_{ii} = 0$ for all $i = 1, \cdots, m$,
- compute M^* for the the Bellman chain on S_n^m of transition cost M defined by $M_{x,T_{ij}(x)} \stackrel{\text{def}}{=} r_{ij}$ and

$$T_{ij}(x_1, \cdots, x_m) \stackrel{\text{def}}{=} (x_1, \cdots, x_i - 1, \cdots, x_j + 1, \cdots, x_m) ,$$

  for $i, j = 1, \cdots, m$.
- The operator T_{ij} corresponds to the transportation of a car from the parking i to the parking j.
- If $r_{ii} = e$ for all $i = 1, \cdots, m$ (the absence of transportation costs nothing) the previous problem corresponds to the computation of the largest invariant cost c satisfying $c = cM$, and $c_x = e$.

3.2. Solution to the m-parkings transportation problem

THEOREM 11. *The optimal value of the transportation problem is :*

$$M^*_{xy} = \mathcal{P}^*_{xy}(M) = \inf_{\substack{\phi \geq 0 \\ \mathcal{J}\phi = y-x}} \phi.r^* \; .$$

where $\mathcal{J}$ *the incidence matrix nodes-arcs of the complete graph and* $\phi.r = \sum_{i,j} \phi_{ij} r_{ij}$.

We have for all y *and* x *such that* $x_j \leq y_j$ *for* $j \neq i$

$$M^*_{xy} = \bigotimes_{j, j \neq i} (r^*_{ij})^{(y_j - x_j)} \; ,$$

and for all x *and* y *satisfying* $y_j \leq x_j$ *for* $j \neq i$

$$M^*_{xy} = \bigotimes_{j, j \neq i} (r^*_{ji})^{(x_j - y_j)} \; .$$

3.3. EXAMPLE

Transportation system, Figure (2), with 3 parkings and 6 cars, and transportation costs :

$$r = \begin{pmatrix} 0 & 1 & +\infty \\ +\infty & 0 & 1 \\ 1 & +\infty & 0 \end{pmatrix} = \begin{pmatrix} e & 1 & \epsilon \\ \epsilon & e & 1 \\ 1 & \epsilon & e \end{pmatrix} .$$

We have :

$$r^* = \begin{pmatrix} e & 1 & 2 \\ 2 & e & 1 \\ 1 & 2 & e \end{pmatrix} .$$

$x = (0, 0, 6)$, $y = (2, 3, 1)$,

$$M^*_{xy} = (r^*_{31})^2 (r^*_{32})^3 = 2 \times 1 + 3 \times 2 = 8 .$$

3.4. AGGREGATION

- Given $\mathcal{X} = \overline{\mathbb{R}}^n_{\min}$, $\mathcal{Y} = \overline{\mathbb{R}}^p_{\min}$ and $C : \mathcal{X} \to \mathcal{Y}$ a linear map. We say that $A : \mathcal{X} \to \mathcal{X}$ is aggregable with C if there exists A_C such that
$$CA = A_C C.$$
- If A is aggregable by C and $X_{n+1} = AX_n$ then $Y_n \triangleq CX_n$ satisties
$$Y_{n+1} = A_C Y_n.$$
- Given a partition $\mathcal{U} = \left\{J_1, \dots, J_p\right\}$ of the state space $F = \{1, \dots, n\}$, the characteristic matrix of the partition $\mathcal{U}$ is
$$U_{iJ} = \begin{cases} e & \text{si } i \in J, \\ \varepsilon & \text{si } i \notin J, \end{cases} \quad \forall i \in F,\ \forall J \in \mathcal{U}\ .$$
- A is aggregable with U^t we say lumpable iff
$$\bigoplus_{k \in K} a_{kj} = \overline{a}_{KJ},\ \forall j \in J,\ \forall J, K \in \mathcal{U}\ .$$

4. Input-Output Max-Plus Linear Systems

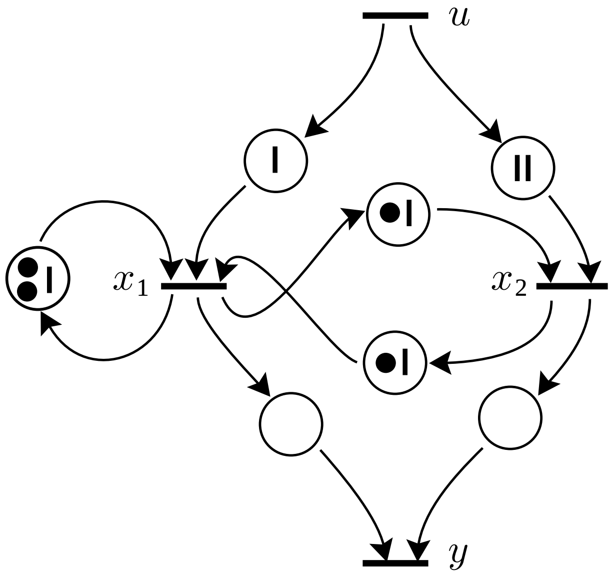

Figure 3. Event Graph

$$\begin{cases} x_k^1 = \max(1 + x_{k-2}^1, 1 + x_{k-1}^2, 1 + u_k) \\ x_k^2 = \max(1 + x_{k-1}^1, 2 + u_k) \\ y_k = \max(x_k^1, x_k^2) \end{cases} \qquad \begin{cases} x_t^1 = \min(x_{t-1}^1 + 2, x_{t-1}^2 + 1, u_{t-1}) \\ x_t^2 = \min(x_{t-1}^1 + 1, u_{t-2}) \\ y_t = \min(x_t^1, x_t^2) \end{cases}$$

$$D = \bigoplus_{k\in\mathbb{Z}} d_k \gamma^k,\ c_k \in \overline{\mathbb{Z}}_{\max}\ .\quad C = \bigoplus_{t\in\mathbb{Z}} c_t \delta^t,\ d_t \in \overline{\mathbb{Z}}_{\min}\ .$$

$$\gamma : (d_k)_{k\in\mathbb{Z}} \mapsto (d_{k-1})_{k\in\mathbb{Z}}\ .\quad \delta : (c_t)_{t\in\mathbb{Z}} \to (c_{t-1})_{t\in\mathbb{Z}}\ .$$

$$\begin{cases} X = \gamma A X \oplus BU\ , \\ Y = CX\ . \end{cases} \qquad \begin{cases} X = \delta \tilde{A} X \oplus \tilde{B} U\ , \\ Y = \tilde{C} X\ . \end{cases}$$

$$Y = C\,(\gamma A)^*\, BU\ .\quad Y = \tilde{C}\left(\delta \tilde{A}\right)^* \tilde{B} U\ .$$

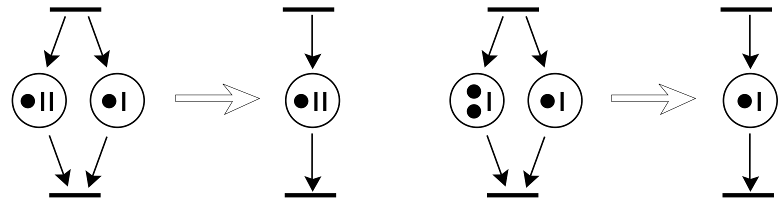

FIGURE 4. Event graph simplification.

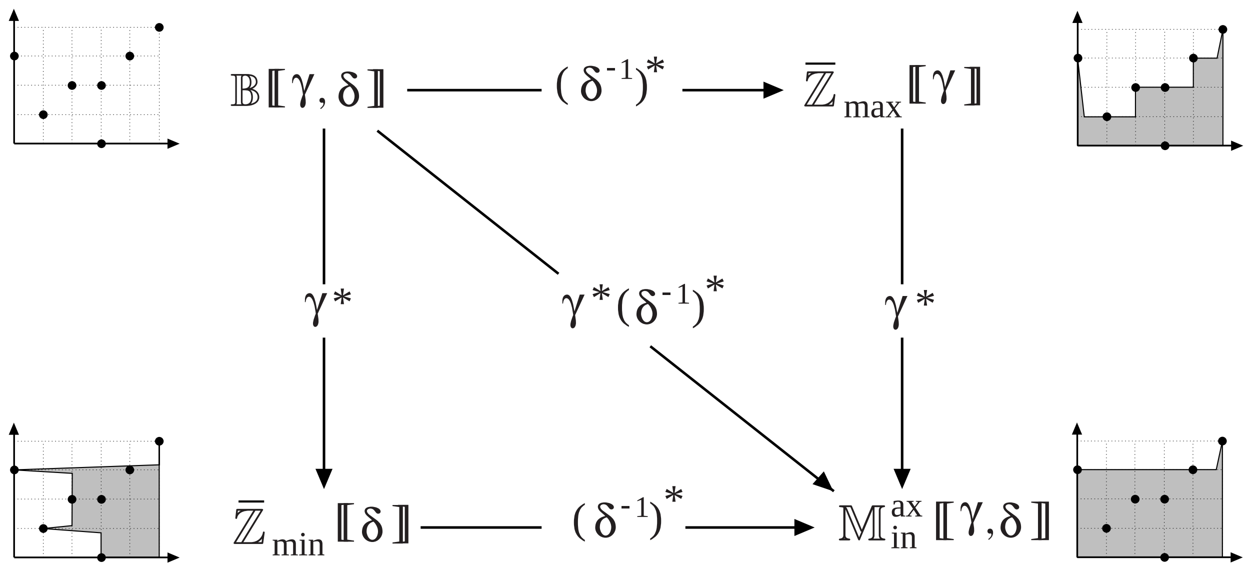

FIGURE 5. Modellings

$$\begin{cases} X = AX \oplus BU \ , \\ Y = CX \ , \end{cases} \qquad A = \begin{bmatrix} \gamma^2\delta & \gamma\delta \\ \gamma\delta & \varepsilon \end{bmatrix}, \ B = \begin{bmatrix} \delta \\ \delta^2 \end{bmatrix}, \quad C = \begin{bmatrix} e & e \end{bmatrix} \ .$$

$$Y = CA^*BU = \delta^2 (\gamma\delta)^* U \ .$$

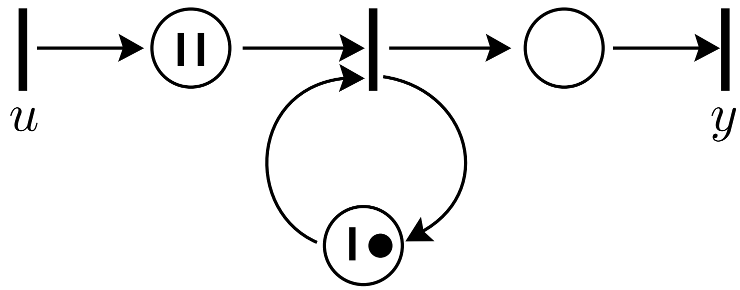

FIGURE 6. Equivalent system 3.

4.2. Rational Series.

$S \in \mathbb{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$ is :

1. rational if it belongs to the closure $\{\varepsilon, e, \gamma, \delta\}$ with respect of finite number of operations $\oplus$, $\otimes$ and $*$;
2. realizable if it can be written :

$$S = C\,(\gamma A_1 \oplus \delta A_2)^*\, B \;,$$

   with C, A_1, A_2, B boolean ;
3. periodic if it exists p, q polynomials and m monomial such that :

$$S = p \oplus q m^* \;.$$

Theorem 12.

Rational $\Leftrightarrow$ Realizable $\Leftrightarrow$ Periodic.

Troughput of an event graph. $A(\gamma, \delta)$ irreducible,

$$\lambda = \max_{m \in C \in \mathcal{C}} \frac{m_\delta}{m_\gamma} , \quad m = \gamma^{m_\gamma} \delta^{m_\delta} .$$

Feedback design.

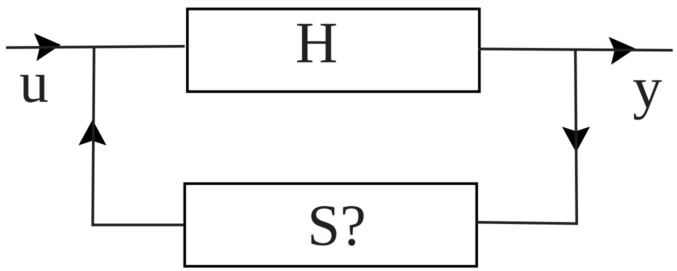

FIGURE 7. Feedback.

$$Y = H(U \oplus SY) = (HS)^* HU .$$

Latest entrance time to achieve an objective.

$$Z = CA^*BU \leqslant Y , \quad U = CA^*B \backslash Y , \quad \begin{cases} \xi = A \backslash \xi \wedge C \backslash Y , \\ Y = B \backslash \xi . \end{cases}$$

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